Geometry of amplitudes: cuts, singularities, vectors fields and other applications of computational algebraic geometry in theoretical physics.

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Modern theoretical physics often has problems with complicated algebraic constraints

Scattering Amplitudes

- Integration-by-Parts reduction for Feynman integrals
- Differential equation for Feynman integral

A lot of relations of polynomial or rational functions

Integrability

- Bethe Ansatz Equation

Usually there are several parameters,
and we need analytic result in these parameters.

New methods of computational method needed

Computational Algebraic Geometry

Singular, Macaulay2, Magma, CoCoA5
Fgb, …

Integration-by-Parts (IBP) reduction
New approaches

Gluza, Kajda, Kosower 1009.0472

Smaller linear system, even no linear system

I. Baikov representation without "doubled propagators"

$$\int \frac{d^D l_1}{i\pi^{D/2}} \ldots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \ldots D_m^{\alpha_m} D_{m+1}^{\alpha_{m+1}} \ldots D_k^{\alpha_k}}, \quad \begin{cases} \alpha_i \leq 1, & 1 \leq i \leq m \\ \alpha_i \leq 0, & m < i \leq k \end{cases}$$

Smaller set of target integrals
more physical

review
Roman Lee, arXiv:1405.5616

II. Feynman representation without "numerators"

$$\int \frac{d^D l_1}{i\pi^{D/2}} \ldots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \ldots D_m^{\alpha_m} D_{m+1}^{\alpha_{m+1}} \ldots D_k^{\alpha_k}}, \quad \begin{cases} \alpha_i \geq 0, & 1 \leq i \leq m \\ \alpha_i = 0, & m < i \leq k \end{cases}$$

Smaller set of target integrals
also useful for cases

Modern IBP reduction:
Only work with the chosen smaller integral set
Hopefully to get a much smaller linear system

Not arbitrary IBP,
but some IBPs satisfying
algebraic constraints

Computational Algebraic Geometry!

Baikov representation without "doubled propagators"

Ita 1510.05626
Larsen and YZ, 1511.01071
Georgoudis, Larsen, YZ, 1612.04252
Zeng 1702.02335

When $k = LE + L(L+1)/2$, (E is the number of independent legs), the Baikov rep. is

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}} \propto \prod_{1\le i\le L+E, \max\{i,E+1\}\le j\le L+E} \left(\int dx_{ij}\right) \det(S)^{\frac{D-L-E-1}{2}} \frac{1}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}}$$

Linear function of x's

$$\propto \prod_{i=1}^{k} \left(\int dz_i\right) \det(S)^{\frac{D-L-E-1}{2}} \frac{1}{z_1^{\alpha_1} \ldots z_k^{\alpha_k}}$$

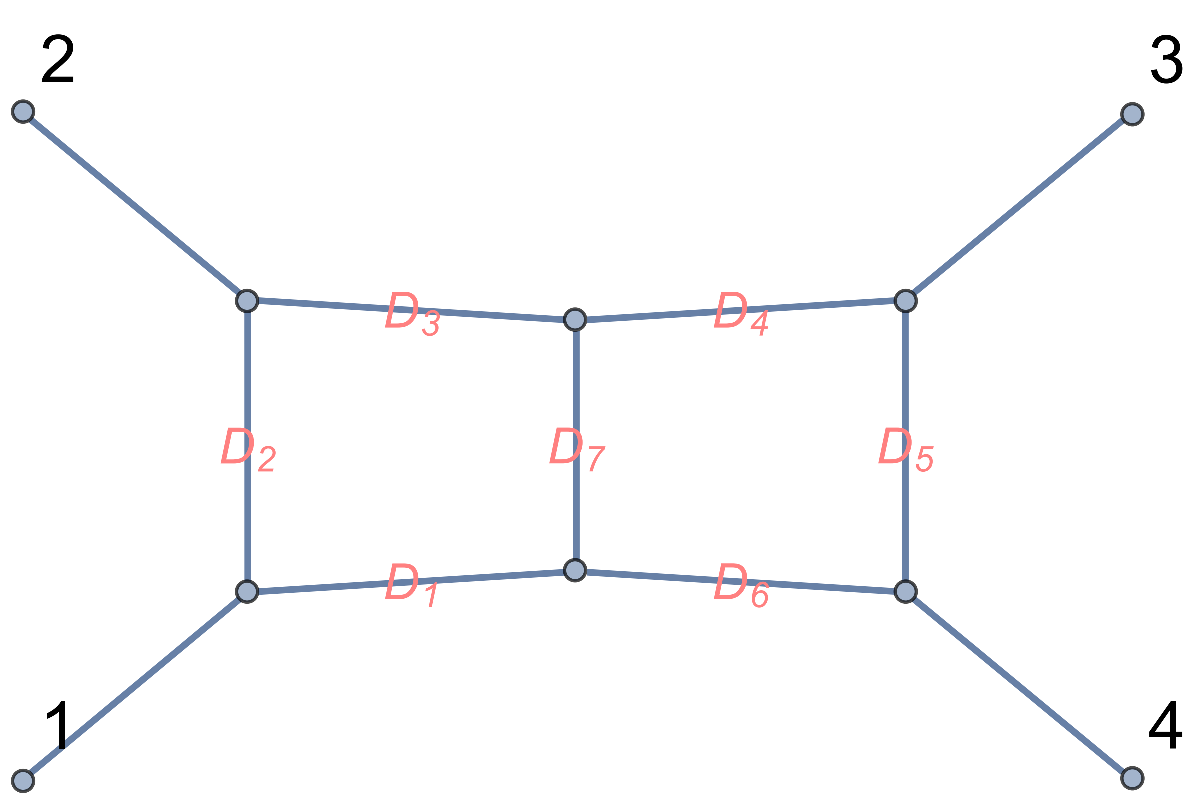
$z_i \equiv D_i$

Baikov rep.

where $\{v_1, \ldots v_{L+E}\} \equiv \{k_1, \ldots k_E, l_1, \ldots l_L\}$ and $x_{ij} \equiv v_i \cdot v_j$. S is a $(L+E) \times (L+E)$ matrix with $S_{ij} = x_{ij}$ (Gram matrix).

Griffiths-Dwork

$L = 2$, $E = 3$, $m = 7$ and $k = 9$. $\{v_1, \ldots, v_5\} \equiv \{k_1, k_2, k_4, l_1, l_2\}$.



$$S = \begin{pmatrix} 0 & \frac{s}{2} & \frac{t}{2} & x_{11} & x_{21} \\ \frac{s}{2} & 0 & \frac{1}{2}(-s-t) & x_{12} & x_{22} \\ \frac{t}{2} & \frac{1}{2}(-s-t) & 0 & x_{13} & x_{23} \\ x_{11} & x_{12} & x_{13} & x_{44} & x_{45} \\ x_{21} & x_{22} & x_{23} & x_{45} & x_{55} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{s}{2} & \frac{t}{2} & \frac{z_1}{2} - \frac{z_2}{2} & \frac{z_9}{2} - \frac{z_6}{2} \\ \frac{s}{2} & 0 & \frac{1}{2}(-s-t) & \frac{1}{2}(s - z_3) + \frac{z_2}{2} & \frac{1}{2}(z_4 - s) - \frac{z_9}{2} \\ \frac{t}{2} & \frac{1}{2}(-s-t) & 0 & \frac{z_8}{2} - \frac{z_1}{2} & \frac{z_6}{2} - \frac{z_5}{2} \\ \frac{z_1}{2} - \frac{z_2}{2} & \frac{1}{2}(s - z_3) + \frac{z_2}{2} & \frac{z_8}{2} - \frac{z_1}{2} & z_1 & -\frac{z_1}{2} - \frac{z_6}{2} + \frac{z_7}{2} \\ \frac{z_9}{2} - \frac{z_6}{2} & \frac{1}{2}(z_4 - s) - \frac{z_9}{2} & \frac{z_6}{2} - \frac{z_5}{2} & -\frac{z_1}{2} - \frac{z_6}{2} + \frac{z_7}{2} & z_6 \end{pmatrix}$$

Baikov representation without “doubled propagators”

$R = \mathbb{Q}(\text{parameters})[z_1, \ldots z_k]$

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \ldots D_m^{\alpha_m} D_{m+1}^{\alpha_{m+1}} \ldots D_k^{\alpha_k}}, \quad \begin{cases} \alpha_i \leq 1, & 1 \leq i \leq m \\ \alpha_i \leq 0, & m < i \leq k \end{cases}$$

Just consider IBPs

$$0 = \left(\prod_{i=1}^{k} \int dz_i \right) \sum_{j=1}^{k} \frac{\partial}{\partial z_j} \left(a_j(z) \det(S)^{\frac{D-L-E-1}{2}} \frac{1}{z_1 \ldots z_m} \right)$$

Polynomials!

Further require $F \equiv \det(S)$

“Affine varieties and Lie algebras of vector fields”
Hauser, Müller 1993

1. no shifted exponent: $\sum_{j=1}^{k} a_j(z) \frac{\partial F}{\partial z_j} + \beta(z) F = 0$ These $(a_1(z), \ldots a_k(z))$ form a module $M_1 \subset R^k$.

2. no doubled propagator: $a_i(z) \in \langle z_i \rangle, \quad 1 \leq i \leq m$ These $(a_1(z), \ldots a_k(z))$ form a module $M_2 \subset R^k$.

Both M_1 and M_2 are pretty simple ...

Y.Z. 1612.02249 $M_1 \cap M_2$ Intersection of two modules

Logarithmic vector field

$$\sum_{j=1}^{k} a_j(z)\frac{\partial F}{\partial z_j} + \beta(z)F = 0$$

$\mathcal{D}$ is the divisor (hyper-surface) $F = 0$,

$$\alpha_i(z)\frac{\partial}{\partial z_i} \in \mathrm{Der}(\log \mathcal{D}),$$

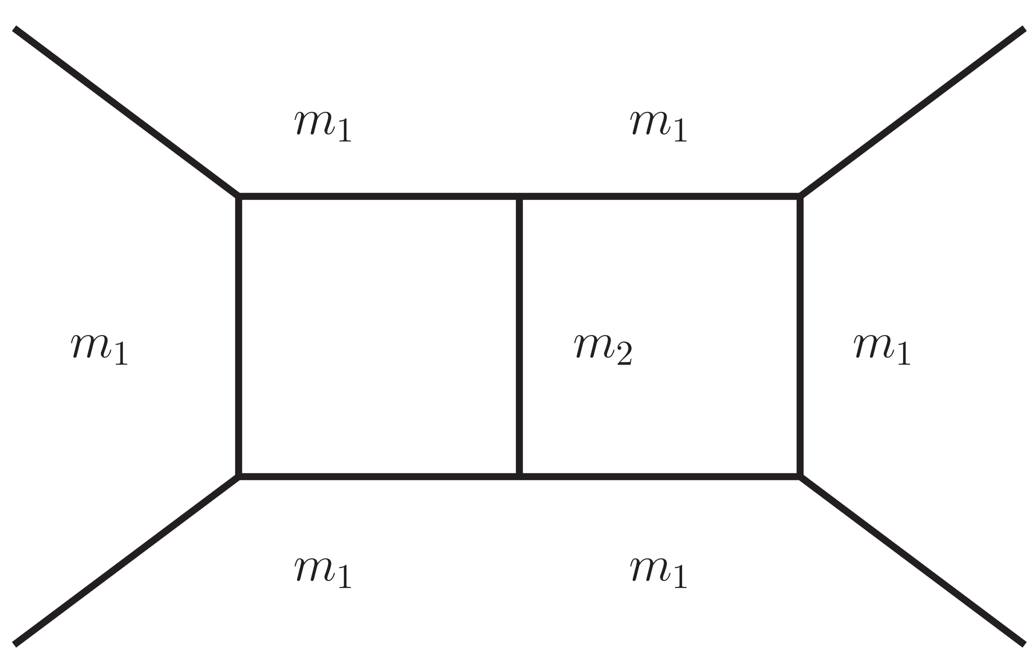
Geometry of the hyper-surface

$J = \langle \frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_k}, F\rangle$ is the singular ideal. If $J = \langle 1\rangle$, then $\mathcal{D}$ is smooth. In the smooth case,

- $\mathrm{Der}(\log D)$ is a free module.
- $\mathrm{Der}(\log D)$ is generated by "principal" syzygies of $\{\frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_k}, F\}$.

Quillen–Suslin theorem

Example

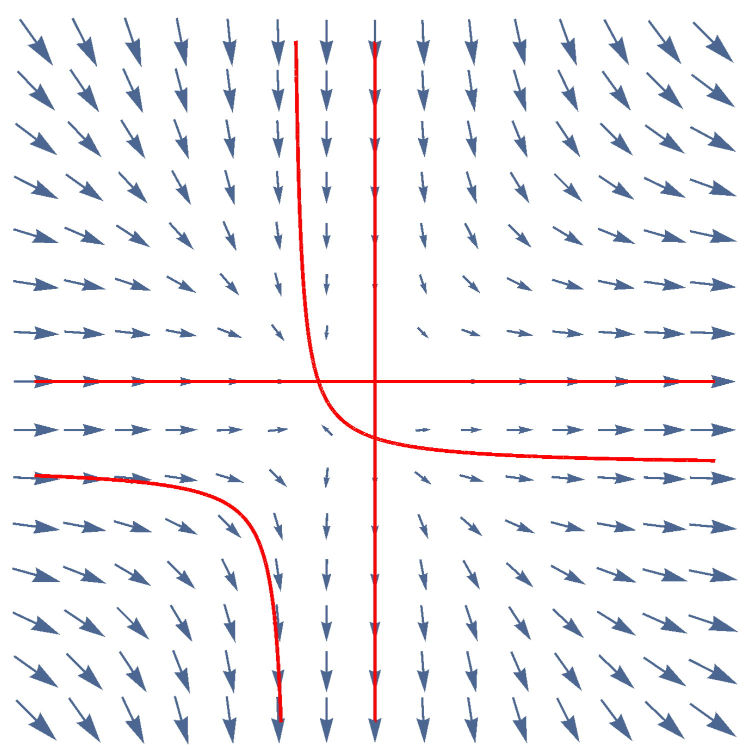


Baikov

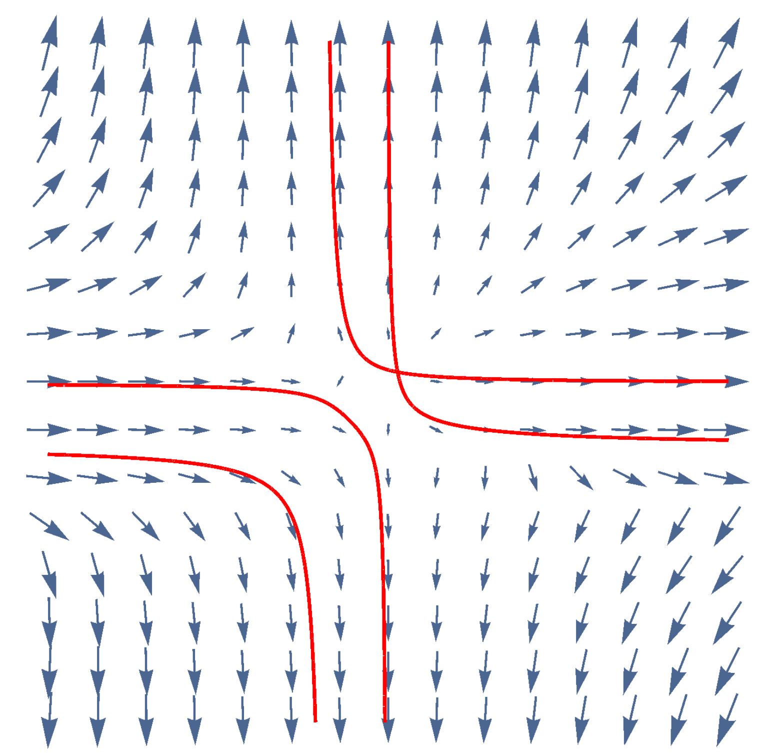
$$I^D_{\mathrm{dbox}}|_{\mathrm{cut}} \quad \propto \quad \int\int dz_8 dz_9 F(z_8, z_9)^{\frac{D-6}{2}} N(z_8, z_9)$$

		$F(x,y) = 0$
Case I	$m_1 = m_2 = 0$	reducible curve: two lines plus one conic
Case II	$m_1 \neq 0, m_2 = 0$	deformed elliptic curve
Case III	$m_1 \neq 0, m_2 \neq 0$	elliptic curve

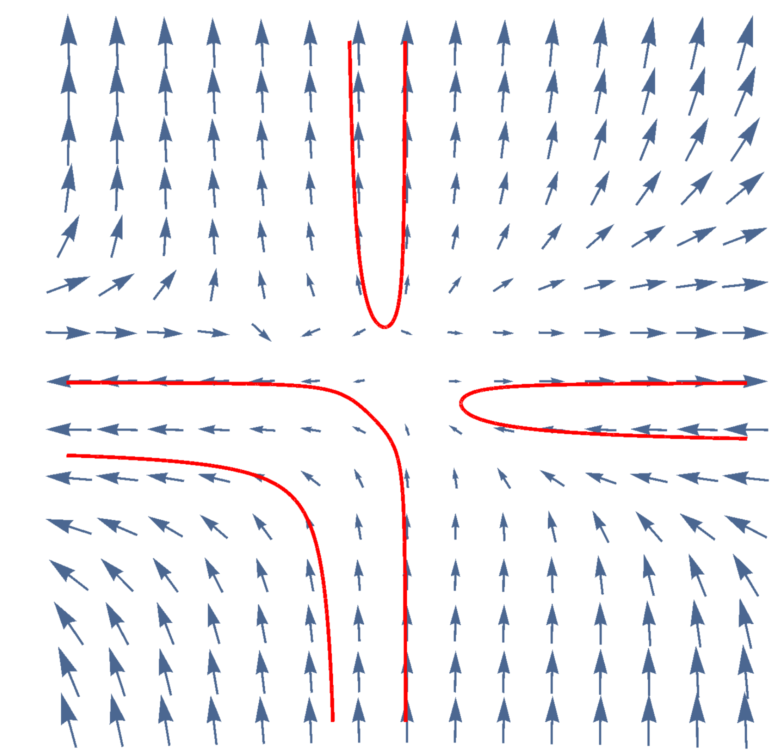
Logarithmic vector field



Case I, three singular points



Case II, one singular point



Case III, no singular point

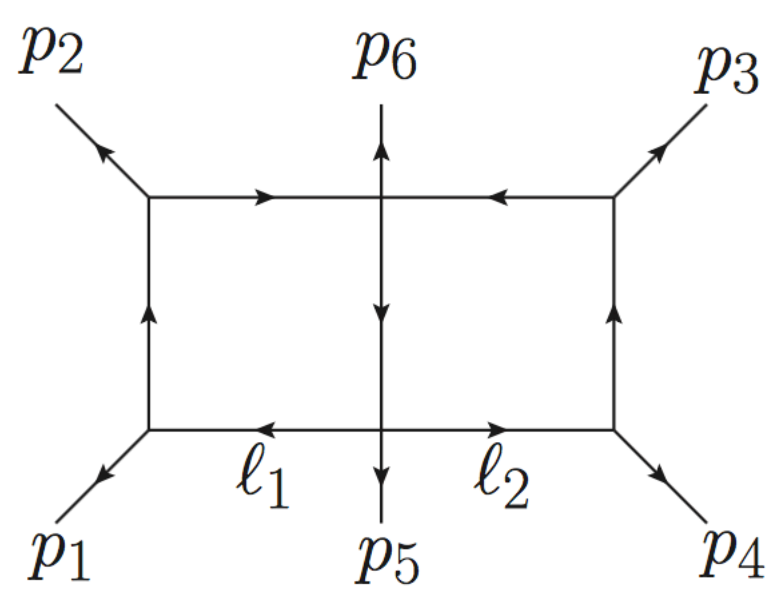
↑ Principal syzygy only

In general, syzygies can be computed by Groebner basis + Schreyer theorem

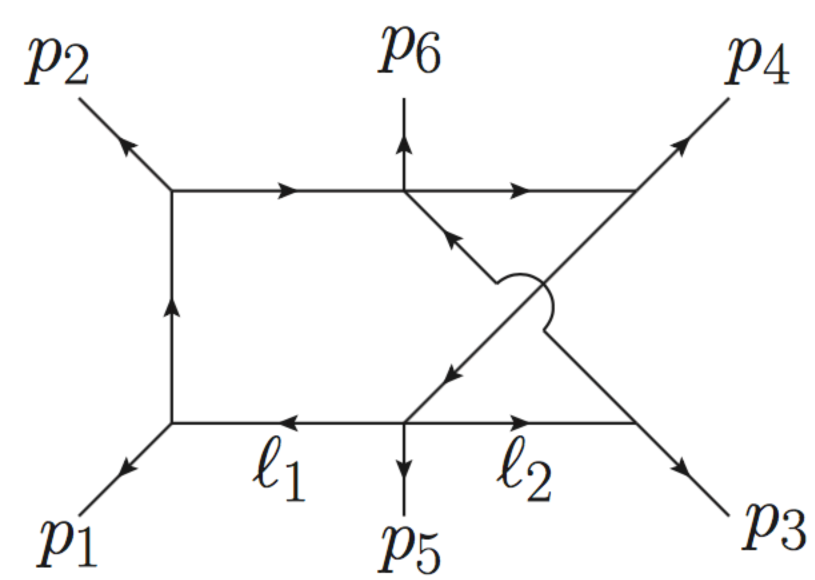
Genus of cut surface

planar curve:

Geometric genus = Arithmetic genus - multiplicity of singular points



g=1



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Huang, YZ 1304. 2013

All massive diagram

Numeric algebraic geometry method

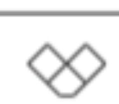
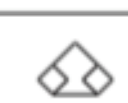
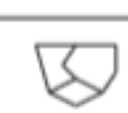
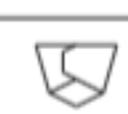
Diagram													
Genus	5	5	9	9	9	13	13	13	13	17	21	21	33

Diagram								
Genus	9	13	17	21	29	33	45	55

Hauenstein, Huang, Mehta, YZ 1408.3355

Baikov representation without "doubled propagators"

$$\sum_{j=1}^{k} a_j(z)\frac{\partial F}{\partial z_j} + \beta(z)F = 0$$

- syzygy for the $\{\frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_k}, F\}$
- Ann(F^s), annihilator of F^s in Weyl algebra.

If F is a determinant matrix whose elements are free variables, this kind of syzygy module is simple. (Roman Lee's idea) http://mathsketches.blogspot.ru/2010/07/blog-post.html

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

syzygy generators (Laplace expansion)

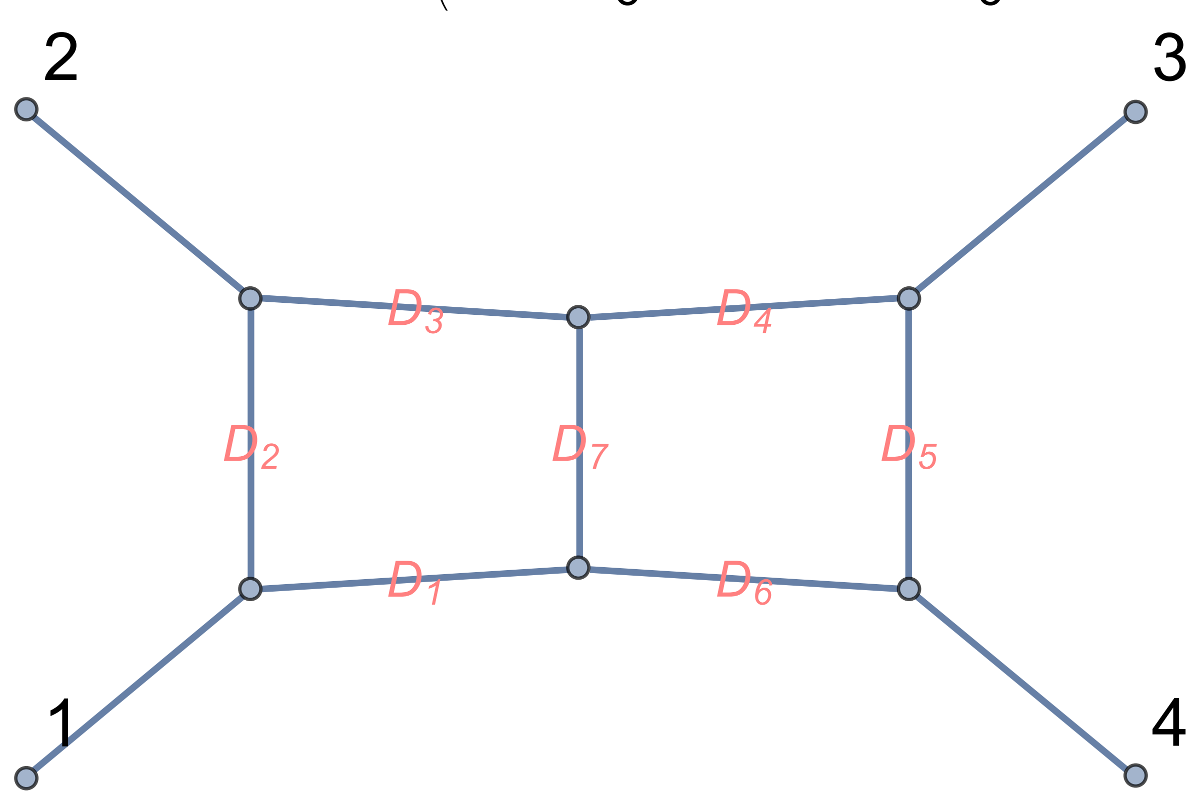
$$\sum_j a_{k,j}\frac{\partial(\det A)}{\partial a_{i,j}} - \delta_{k,i}\cdot\det A = 0$$

to be used in Azurite 2.0.0
Georgoudis, K. Larsen and YZ
to appear fall, 2017

Baikov representation without “doubled propagators”, example I

$\mathbb{Q}(s,t)[z_1,\ldots z_9]$: 2 parameters, 9 variables (Each row is a module generator)

$$M_1 = \begin{pmatrix}
z_1-z_2 & z_1-z_2 & -s+z_1-z_2 & 0 & 0 & 0 & z_1-z_2-z_6+z_9 & t+z_1-z_2 & 0 \\
0 & 0 & 0 & s-z_6+z_9 & -t-z_6+z_9 & -z_6+z_9 & z_1-z_2-z_6+z_9 & 0 & -z_6+z_9 \\
s+z_2-z_3 & z_2-z_3 & z_2-z_3 & 0 & 0 & 0 & z_2-z_3+z_4-z_9 & -t+z_2-z_3 & 0 \\
0 & 0 & 0 & z_4-z_9 & t+z_4-z_9 & -s+z_4-z_9 & z_2-z_3+z_4-z_9 & 0 & z_4-z_9 \\
-z_1+z_8 & -t-z_1+z_8 & s-z_1+z_8 & 0 & 0 & 0 & -z_1-z_5+z_6+z_8 & -z_1+z_8 & 0 \\
0 & 0 & 0 & -s-z_5+z_6 & -z_5+z_6 & -z_5+z_6 & -z_1-z_5+z_6+z_8 & 0 & t-z_5+z_6 \\
2z_1 & z_1+z_2 & -s+z_1+z_3 & 0 & 0 & 0 & z_1-z_6+z_7 & z_1+z_8 & 0 \\
0 & 0 & 0 & s-z_3-z_6+z_7 & -z_6+z_7-z_8 & -z_1-z_6+z_7 & z_1-z_6+z_7 & 0 & -z_2-z_6+z_7 \\
-z_1-z_6+z_7 & -z_1+z_7-z_9 & s-z_1-z_4+z_7 & 0 & 0 & 0 & -z_1+z_6+z_7 & -z_1-z_5+z_7 & 0 \\
0 & 0 & 0 & -s+z_4+z_6 & z_5+z_6 & 2z_6 & -z_1+z_6+z_7 & 0 & z_6+z_9
\end{pmatrix}$$



$$M_2 = \begin{pmatrix}
z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & z_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & z_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & z_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & z_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & z_7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$M_1 \cap M_2$ found in ~4 seconds, with Singular 4.1.0, intersect(M1,M2,”std”)

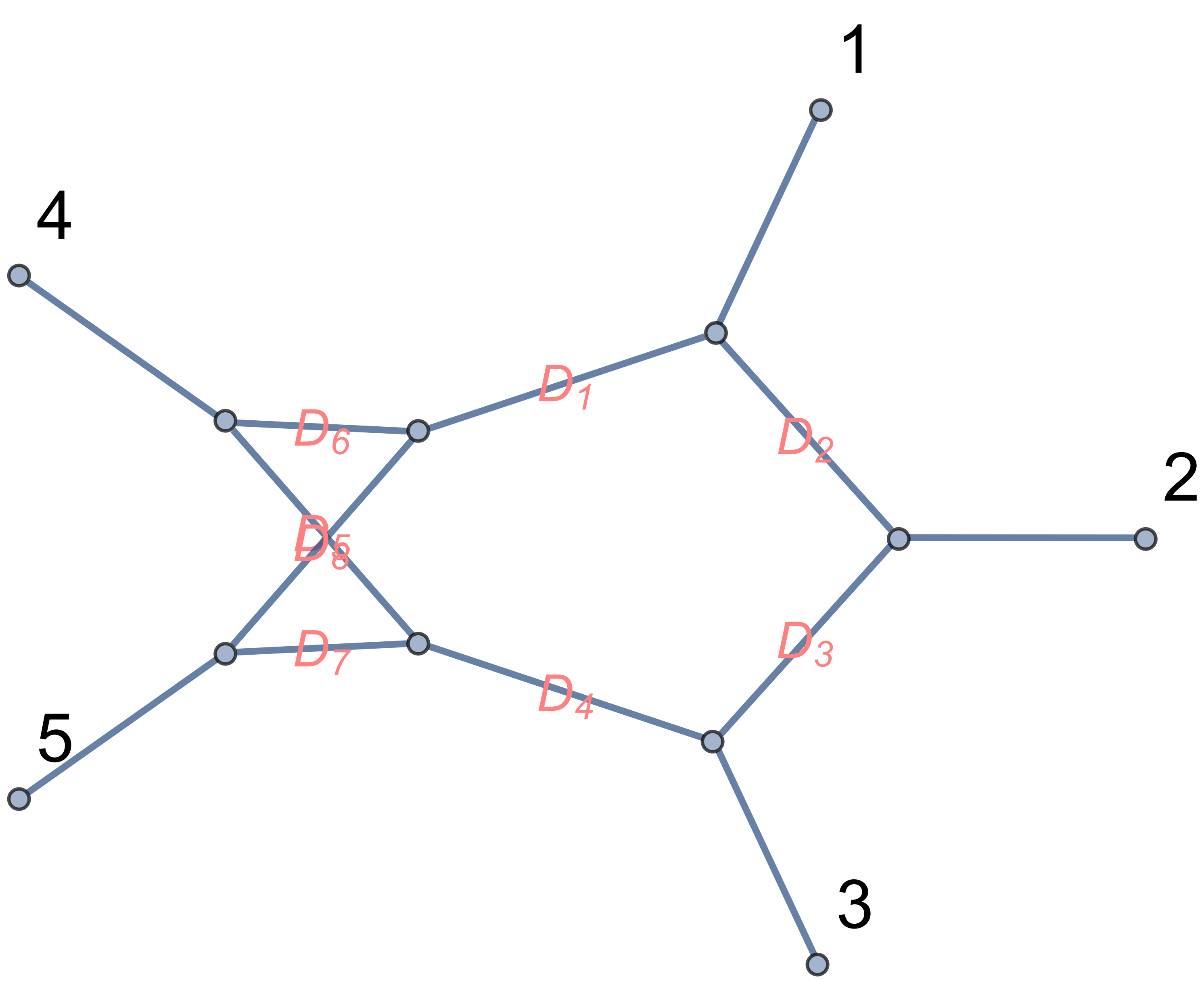
Complete IBP reduction in 39 seconds, (Singular + Mathematica)

Complete IBP reduction in 211 seconds, for one-massive-double-box

Baikov representation without “doubled propagators”, example II

$\mathbb{Q}(s_{12}, s_{13}, s_{14}, s_{23}, s_{24})[z_1, \ldots z_{11}]$: 5 parameters, 11 variables

$$M_1 = \left(\begin{array}{ccccccccccc}
z_1-z_2 & z_1-z_2 & -s_{12}+z_1-z_2 & -s_{12}-s_{13}+z_1-z_2 & s_{14}+z_1-z_2-z_8+z_{10} & z_1-z_2-z_8+z_{10} & 0 & 0 & -s_{12}-s_{13}-s_{14}+z_1-z_2 & 0 & 0 \\
0 & 0 & 0 & 0 & s_{14}+z_1-z_2-z_8+z_{10} & z_1-z_2-z_8+z_{10} & s_{12}+s_{13}+s_{14}-z_8+z_{10} & -z_8+z_{10} & 0 & -z_8+z_{10} & s_{12}-z_8+z_{10} \\
s_{12}+z_2-z_3 & z_2-z_3 & z_2-z_3 & -s_{23}+z_2-z_3 & s_{12}+s_{24}+z_2-z_3-z_8+z_{11} & s_{12}+z_2-z_3-z_8+z_{11} & 0 & 0 & -s_{23}-s_{24}+z_2-z_3 & 0 & 0 \\
0 & 0 & 0 & 0 & s_{12}+s_{24}+z_2-z_3-z_8+z_{11} & s_{12}+z_2-z_3-z_8+z_{11} & s_{12}+s_{23}+s_{24}-z_8+z_{11} & -z_8+z_{11} & 0 & s_{12}-z_8+z_{11} & -z_8+z_{11} \\
s_{13}+s_{23}+z_3-z_4 & s_{23}+z_3-z_4 & z_3-z_4 & z_3-z_4 & -2s_{12}-s_{13}-s_{14}-s_{23}-s_{24}+z_3-z_5+z_6+z_7+z_8-z_9-z_{10}-z_{11} & -s_{12}+z_3-z_5+z_6+z_7+z_8-z_9-z_{10}-z_{11} & 0 & 0 & s_{12}+s_{13}+s_{14}+s_{23}+s_{24}+z_3-z_4 & 0 & 0 \\
0 & 0 & 0 & 0 & -2s_{12}-s_{13}-s_{14}-s_{23}-s_{24}+z_3-z_5+z_6+z_7+z_8-z_9-z_{10}-z_{11} & -s_{12}+z_3-z_5+z_6+z_7+z_8-z_9-z_{10}-z_{11} & -2s_{12}-s_{13}-s_{14}-s_{23}-s_{24}+z_4-z_5+z_6+z_7+z_8-z_9-z_{10}-z_{11} & -s_{12}-s_{13}-s_{23}+z_4-z_5+z_6+z_7+z_8-z_9-z_{10}-z_{11} & 0 & -s_{12}-s_{23}+z_4-z_5+z_6+z_7+z_8-z_9-z_{10}-z_{11} & -s_{12}-s_{13}+z_4-z_5+z_6+z_7+z_8-z_9-z_{10}-z_{11} \\
-s_{12}-s_{13}-s_{23}+z_4-z_9 & -s_{12}-s_{13}-s_{14}-s_{23}+z_4-z_9 & -s_{12}-s_{13}-s_{14}-s_{23}-s_{24}+z_4-z_9 & z_4-z_9 & z_5-z_6 & z_5-z_6 & 0 & 0 & z_4-z_9 & 0 & 0 \\
0 & 0 & 0 & 0 & z_5-z_6 & z_5-z_6 & -z_4+z_5-z_6+z_9 & s_{12}+s_{13}+s_{23}-z_4+z_5-z_6+z_9 & 0 & s_{12}+s_{13}+s_{14}+s_{23}-z_4+z_5-z_6+z_9 & s_{12}+s_{13}+s_{23}+s_{24}-z_4+z_5-z_6+z_9 \\
2z_1 & z_1+z_2 & -s_{12}+z_1+z_3 & -s_{12}-s_{13}-s_{23}+z_1+z_4 & -s_{12}-s_{13}-s_{23}+z_1+z_4+z_6-z_8-z_9 & z_1+z_6-z_8 & 0 & 0 & z_1+z_9 & 0 & 0 \\
0 & 0 & 0 & 0 & -s_{12}-s_{13}-s_{23}+z_1+z_4+z_6-z_8-z_9 & z_1+z_6-z_8 & z_6-z_8-z_9 & -z_1+z_6-z_8 & 0 & -z_2+z_6-z_8 & s_{12}-z_1+z_2-z_3+z_6-z_8 \\
-z_1+z_6-z_8 & -z_1+z_6-z_{10} & -z_1+z_6+z_8-z_{10}-z_{11} & s_{12}+s_{13}+s_{23}-z_1-z_4+z_5-z_7+z_9 & s_{12}+s_{13}+s_{23}-z_1-z_4+z_5+z_8+z_9 & -z_1+z_6+z_8 & 0 & 0 & -z_1+z_6-z_7 & 0 & 0 \\
0 & 0 & 0 & 0 & s_{12}+s_{13}+s_{23}-z_1-z_4+z_5+z_8+z_9 & -z_1+z_6+z_8 & z_7+z_8 & 2z_8 & 0 & z_8+z_{10} & z_8+z_{11}
\end{array}\right)$$



$$M_2 = \begin{pmatrix}
z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & z_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & z_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & z_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & z_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & z_7 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & z_8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$M_1 \cap M_2 = ?$$

Open problem for scattering amplitudes

FIRE, reduze, LiteRed cannot find the IBP reduction for this diagram

Baikov representation without “doubled propagators”, example II

variant $z_3 \to 0, z_6 \to 0, z_7 \to 0$

$\mathbb{Q}(s_{12}, s_{13}, s_{14}, s_{23}, s_{24})[z_1, z_2, z_4, z_5, z_8, z_9, z_{10}, z_{11}]$: 5 parameters, 11-3=8 variables

$$M_1' = \begin{pmatrix}
z_1 - z_2 & z_1 - z_2 & -s_{12} + z_1 - z_2 & -s_{12} - s_{13} + z_1 - z_2 & s_{14} + z_1 - z_2 - z_8 + z_{10} & z_1 - z_2 - z_8 + z_{10} & 0 & 0 & -s_{12} - s_{13} - s_{14} + z_1 - z_2 & 0 & 0 \\
0 & 0 & 0 & 0 & s_{14} + z_1 - z_2 - z_8 + z_{10} & z_1 - z_2 - z_8 + z_{10} & s_{12} + s_{13} + s_{14} - z_8 + z_{10} & -z_8 + z_{10} & 0 & -z_8 + z_{10} & s_{12} - z_8 + z_{10} \\
s_{12} + z_2 & z_2 & z_2 & -s_{23} + z_2 & s_{12} + s_{24} + z_2 - z_8 + z_{11} & s_{12} + z_2 - z_8 + z_{11} & 0 & 0 & -s_{23} - s_{24} + z_2 & 0 & 0 \\
0 & 0 & 0 & 0 & s_{12} + s_{24} + z_2 - z_8 + z_{11} & s_{12} + z_2 - z_8 + z_{11} & s_{12} + s_{23} + s_{24} - z_8 + z_{11} & -z_8 + z_{11} & 0 & s_{12} - z_8 + z_{11} & -z_8 + z_{11} \\
s_{13} + s_{23} - z_4 & s_{23} - z_4 & -z_4 & -z_4 & -2 s_{12} - s_{13} - s_{14} - s_{23} - s_{24} - z_5 + z_8 - z_9 - z_{10} - z_{11} & -s_{12} - z_5 + z_8 - z_9 - z_{10} - z_{11} & 0 & 0 & s_{12} + s_{13} + s_{14} + s_{23} + s_{24} - z_4 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 s_{12} - s_{13} - s_{14} - s_{23} - s_{24} - z_5 + z_8 - z_9 - z_{10} - z_{11} & -s_{12} - z_5 + z_8 - z_9 - z_{10} - z_{11} & -2 s_{12} - s_{13} - s_{14} - s_{23} - s_{24} + z_4 - z_5 + z_8 - z_9 - z_{10} - z_{11} & -s_{12} - s_{13} - s_{23} + z_4 - z_5 + z_8 - z_9 - z_{10} - z_{11} & 0 & -s_{12} - s_{23} + z_4 - z_5 + z_8 - z_9 - z_{10} - z_{11} & -s_{12} - s_{13} + z_4 - z_5 + z_8 - z_9 - z_{10} - z_{11} \\
-s_{12} - s_{13} - s_{23} + z_4 - z_9 & -s_{12} - s_{13} - s_{14} - s_{23} + z_4 - z_9 & -s_{12} - s_{13} - s_{14} - s_{23} - s_{24} + z_4 - z_9 & z_4 - z_9 & z_5 & z_5 & 0 & 0 & z_4 - z_9 & 0 & 0 \\
0 & 0 & 0 & 0 & z_5 & z_5 & -z_4 + z_5 + z_9 & s_{12} + s_{13} + s_{23} - z_4 + z_5 + z_9 & 0 & s_{12} + s_{13} + s_{14} + s_{23} - z_4 + z_5 + z_9 & s_{12} + s_{13} + s_{23} + s_{24} - z_4 + z_5 + z_9 \\
2 z_1 & z_1 + z_2 & -s_{12} + z_1 & -s_{12} - s_{13} - s_{23} + z_1 + z_4 & -s_{12} - s_{13} - s_{23} + z_1 + z_4 - z_8 - z_9 & z_1 - z_8 & 0 & 0 & z_1 + z_9 & 0 & 0 \\
0 & 0 & 0 & 0 & -s_{12} - s_{13} - s_{23} + z_1 + z_4 - z_8 - z_9 & z_1 - z_8 & -z_8 - z_9 & -z_1 - z_8 & 0 & -z_2 - z_8 & s_{12} - z_1 + z_2 - z_8 \\
-z_1 - z_8 & -z_1 - z_{10} & -z_1 + z_8 - z_{10} - z_{11} & s_{12} + s_{13} + s_{23} - z_1 - z_4 + z_5 + z_9 & s_{12} + s_{13} + s_{23} - z_1 - z_4 + z_5 + z_8 + z_9 & -z_1 + z_8 & 0 & 0 & -z_1 & 0 & 0 \\
0 & 0 & 0 & 0 & s_{12} + s_{13} + s_{23} - z_1 - z_4 + z_5 + z_8 + z_9 & -z_1 + z_8 & z_8 & 2 z_8 & 0 & z_8 + z_{10} & z_8 + z_{11}
\end{pmatrix}$$

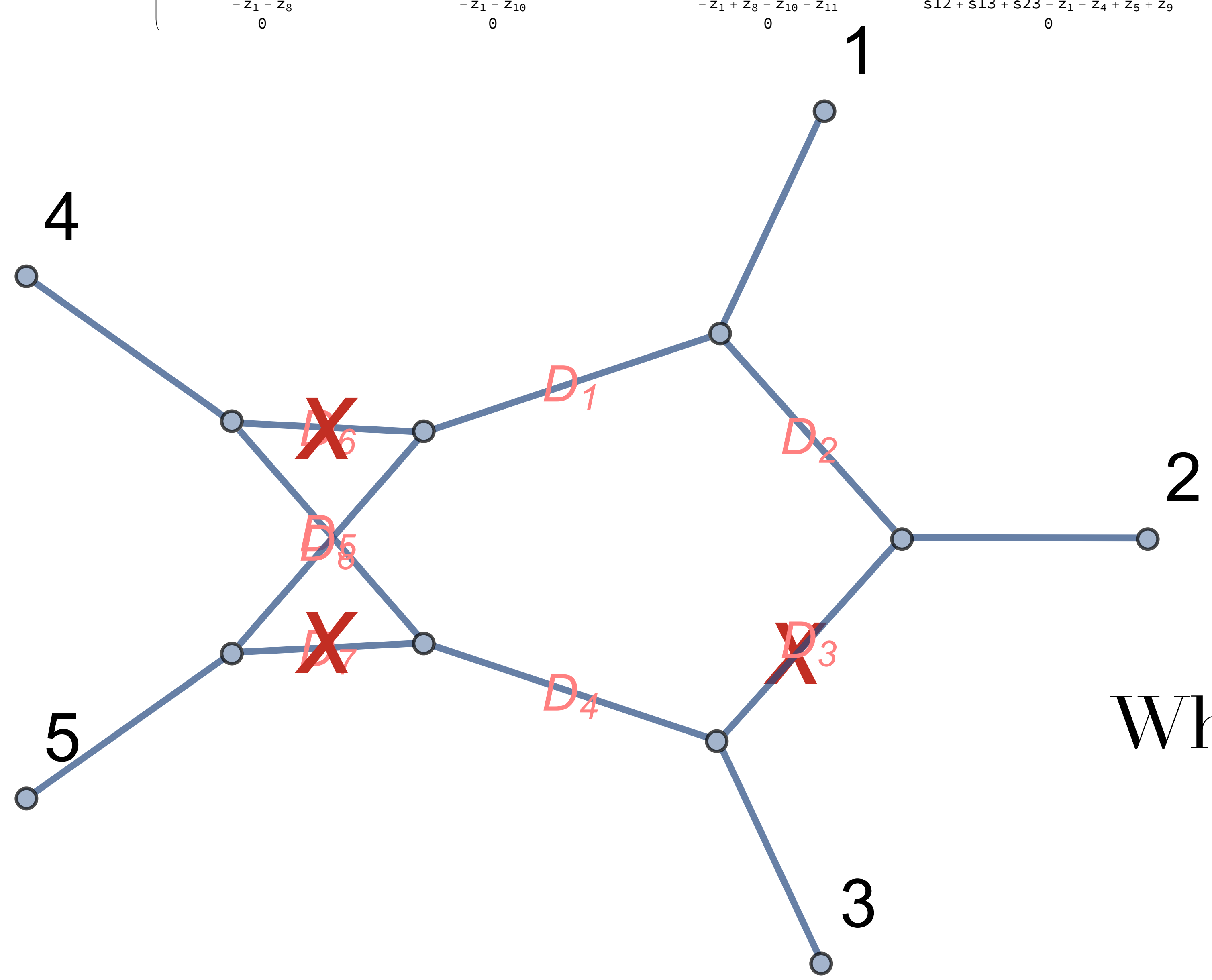


Diagram with a triple cut

$$M_2' = \begin{pmatrix}
z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & z_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & z_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & z_8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$M_1' \cap M_2' = ?$

When the 5 parameters are taken to be numeric (rational number or Z/p), it is a very fast computation for Singular 4.1.0.

We are working the reconstruction of IBP from numeric data (Z/p). (Georgoudis, Larsen and YZ)

symbolic parameter?

Feynman representation without "numerators"

Griffiths-Dwork

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \dots D_m^{\alpha_m} D_{m+1}^{\alpha_{m+1}} \dots D_k^{\alpha_k}}, \quad \begin{cases} \alpha_i \geq 0, & 1 \leq i \leq m \\ \alpha_i = 0, & m < i \leq k \end{cases}$$

Roman Lee, arXiv:1405.5616

When $a_i > 0$, $1 \leq i \leq m$, the Feynman rep. is

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \dots D_m^{\alpha_m}} \propto \left(\prod_{i=1}^{m} \int_0^\infty dz_i \right) G(z)^{-\frac{D}{2}} z_1^{\alpha_1 - 1} \dots z_m^{\alpha_m - 1}$$

(combined) Symanzik polynomial

Looks simpler than the Baikov rep., but less frequently used for the IBP reduction …

IBP $$0 = \left(\prod_{i=1}^{m} \int dz_i \right) \sum_{j=1}^{m} \frac{\partial}{\partial z_j} \left(a_j(z) G(z)^{-\frac{D}{2}} \right)$$

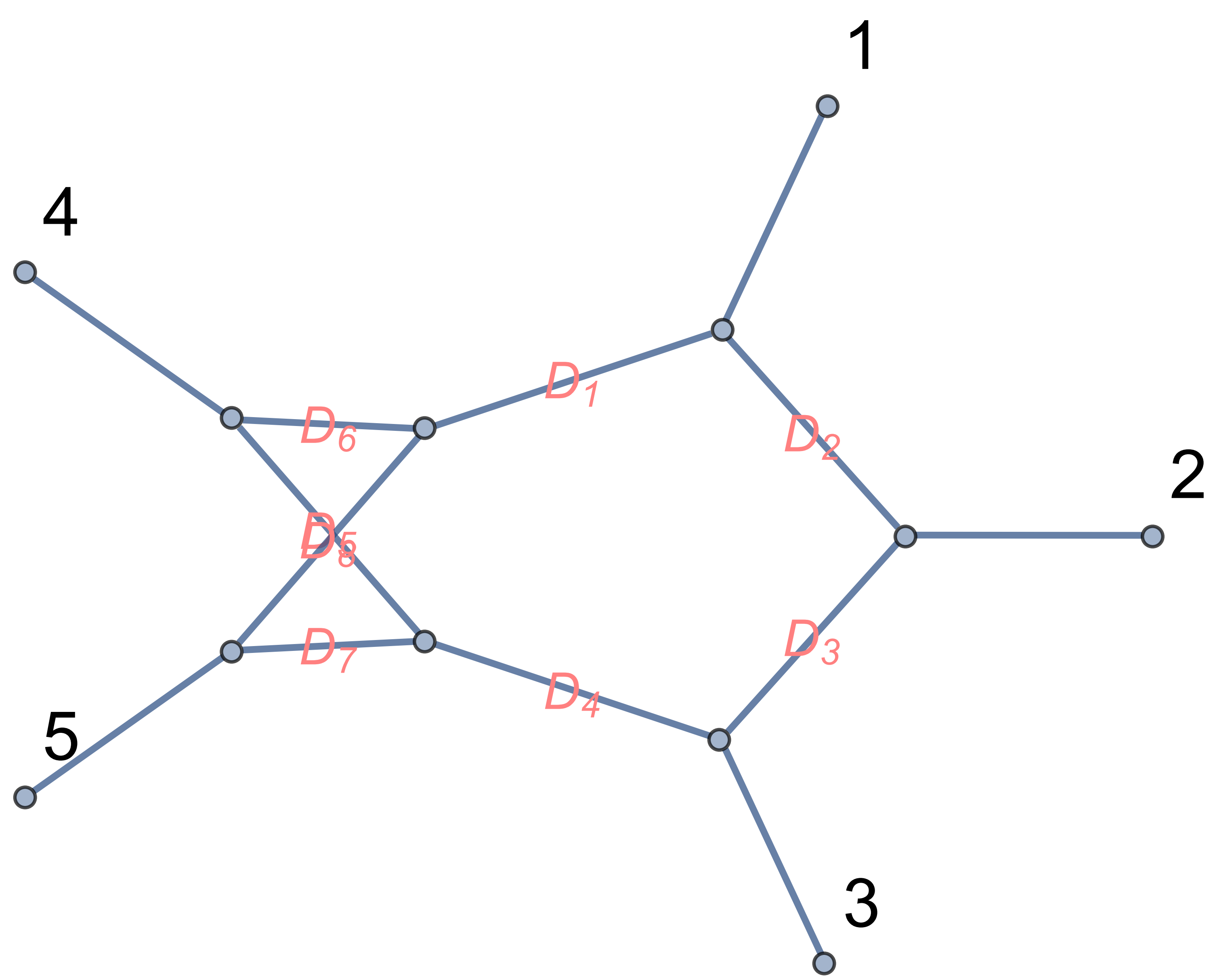
Only one requirement $$\sum_{j=1}^{m} a_j(z) \frac{\partial G}{\partial z_j} + \beta(z) G = 0$$

In general,
G is not simply a matrix determinant…

- syzygy for the $\{\frac{\partial G}{\partial z_1}, \dots, \frac{\partial G}{\partial z_k}, G\}$
- $Ann(G^s)$, annihilator of G^s in Weyl algebra.

Feynman representation without "numerators", example

$\mathbb{Q}(s_{12}, s_{13}, s_{14}, s_{23}, s_{24})[z_1, \ldots z_8]$: 5 parameters, 8 variables



$$\begin{aligned} G = {} & z_1 z_5 + z_2 z_5 + z_3 z_5 - s12\, z_1 z_3 z_5 + z_4 z_5 - s12\, z_1 z_4 z_5 - s13\, z_1 z_4 z_5 - s23\, z_1 z_4 z_5 - s23\, z_2 z_4 z_5 + \\ & z_1 z_6 + z_2 z_6 + z_3 z_6 - s12\, z_1 z_3 z_6 + z_4 z_6 - s12\, z_1 z_4 z_6 - s13\, z_1 z_4 z_6 - s23\, z_1 z_4 z_6 - s23\, z_2 z_4 z_6 + \\ & z_1 z_7 + z_2 z_7 + z_3 z_7 - s12\, z_1 z_3 z_7 + z_4 z_7 - s12\, z_1 z_4 z_7 - s13\, z_1 z_4 z_7 - s23\, z_1 z_4 z_7 - s23\, z_2 z_4 z_7 + \\ & z_5 z_7 - s12\, z_1 z_5 z_7 - s13\, z_1 z_5 z_7 - s23\, z_1 z_5 z_7 - s23\, z_2 z_5 z_7 + z_6 z_7 + s12\, z_2 z_6 z_7 + s13\, z_2 z_6 z_7 + \\ & s14\, z_2 z_6 z_7 + s12\, z_3 z_6 z_7 + s13\, z_3 z_6 z_7 + s14\, z_3 z_6 z_7 + s23\, z_3 z_6 z_7 + s24\, z_3 z_6 z_7 + z_1 z_8 + z_2 z_8 + z_3 z_8 - \\ & s12\, z_1 z_3 z_8 + z_4 z_8 - s12\, z_1 z_4 z_8 - s13\, z_1 z_4 z_8 - s23\, z_1 z_4 z_8 - s23\, z_2 z_4 z_8 + z_5 z_8 - s14\, z_2 z_5 z_8 - \\ & s12\, z_3 z_5 z_8 - s14\, z_3 z_5 z_8 - s24\, z_3 z_5 z_8 + z_6 z_8 - s12\, z_3 z_6 z_8 - s12\, z_4 z_6 z_8 - s13\, z_4 z_6 z_8 - s23\, z_4 z_6 z_8 \end{aligned}$$

$$\sum_{j=1}^{m} a_j(z) \frac{\partial G}{\partial z_j} + \beta(z) G = 0$$

numeric parameter easy

symbolic parameter difficult

Remarks on the remaining linear system

We still need to solve a linear system for IBP,
even if it is much smaller then that from Laporta Algorithm …

1. Fraction-free row reduction (Bareiss, Sasaki-Murao) ?
2. Finite-field reconstruction ?

A Groebner basis division algorithm on Weyl algebra

$$W \bullet R \to R$$

$$\sum_{j=1}^{m} a_j(z) \frac{\partial G}{\partial z_j} + \beta(z) G = 0$$

$$0 = \left(\prod_{i=1}^{m} \int dz_i \right) \sum_{j=1}^{m} \frac{\partial a_j(z)}{\partial z_j} + \frac{D}{2} \beta(z)$$

$$\frac{\partial}{\partial z_j} \cdot a_j(z) + \frac{D}{2} \beta(z)$$

Weyl algebra element

$R \to R/(J \bullet 1)$ IBP reduction

Consider the right ideal J generated by these elements
Stafford theorem

division via Groebner basis?
If successful, it will be much more efficient than linear algebra methods.

Integrability

In classical/quantum physics, some systems can be solved by using symmetries (invariants)

Integrability

Bethe Ansatz for spins chains

$$\left(\frac{u_j+1/2}{u_j-1/2}\right)^L = \prod_{k\neq j}^{S} \frac{u_j-u_k-i}{u_j-u_k+i}, \qquad j=1,2,\dots S$$

$$\prod_{j=1}^{S} \frac{u_j+i/2}{u_j-i/2} = 1$$

u_i's are Bethe roots. Bethe Ansatz equations are symmetric under the permutations of u_i's.

N_r sets of Bethe roots, $\mathbf{u}^{(i)}$'s, $i=1,\dots N_r$.

$$(C^{\bullet\infty})^2 = \frac{1}{N_r}\sum_i \frac{L(l+N)(L+N-l)}{\binom{N+l}{N}\binom{N+L-l}{N}}\left(1-\frac{\gamma(\mathbf{u}^{(i)})}{2}\right)^2 \frac{\mathcal{A}(\mathbf{u}^{(i)})^2}{B(\mathbf{u}^{(i)})^2}$$

"Structure constant"

Symmetric function in u_i's

Integrability

Bethe Ansatz computation via Groebner basis

Yunfeng Jiang, YZ,
to appear Sep. 2017

N_r sets of Bethe roots, $\mathbf{u}^{(i)}$'s, $i = 1, \ldots N_r$.

$$(C^{\bullet\infty})^2 = \frac{1}{N_r} \sum_i \frac{L(l+N)(L+N-l)}{\binom{N+l}{N}\binom{N+L-l}{N}} \left(1 - \frac{\gamma(\mathbf{u}^{(i)})}{2}\right)^2 \frac{\mathcal{A}(\mathbf{u}^{(i)})^2}{B(\mathbf{u}^{(i)})^2}$$

$$\frac{f(\mathbf{u})}{g(\mathbf{u})}$$

$$\sum_i \frac{f(\mathbf{u}^{\{i\}})}{g(\mathbf{u}^{\{i\}})} = \mathrm{tr}\left(M_f M_g^{-1}\right)$$

Companion matrices

When $L = 4$, $S = 4$, $l = 2$, $N = 1$, the Gröbner basis method determines that

$$C^{\bullet\infty} = \frac{16}{63}$$

WITHOUT finding Bethe roots explicitly.

Efficient way to determine number of solutions?

Integrability

Bethe Ansatz computation via Groebner basis

When $S > 4$ the Groebner computation is slow with slimgb or std in Singular 4.1.0

Using symmetry

[illegible]

Groebner basis in u_1, u_2, u_3, u_4

Example $L = 4, S = 4$

$$\{s_1 - 4 s_3,\ -3 s_3 + 80 s_3 s_4,\ -47 - 196 s_2 + 768 s_4 + 5184 s_2 s_4 + 16128 s_4^2,$$
$$-11 - 232 s_2 + 1728 s_3^2 - 10752 s_4 + 16128 s_4^2,\ -14 s_3 + 5 s_2 s_3,$$
$$371 - 1424 s_2 + 1296 s_2^2 - 184800 s_4 + 274176 s_4^2,$$
$$-10957 - 464 s_2 + 2764176 s_4 - 68914944 s_4^2 + 104509440 s_4^3,$$
$$-4 s_3 + u_1 + u_2 + u_3 + u_4,\ s_2 - 4 s_3 u_2 + u_2^2 - 4 s_3 u_3 + u_2 u_3 + u_3^2 - 4 s_3 u_4 + u_2 u_4 + u_3 u_4 + u_4^2,$$
$$-s_3 + s_2 u_3 - 4 s_3 u_3^2 + u_3^3 + s_2 u_4 - 4 s_3 u_3 u_4 + u_3^2 u_4 - 4 s_3 u_4^2 + u_3 u_4^2 + u_4^3,\ s_4 - s_3 u_4 + s_2 u_4^2 - 4 s_3 u_4^3 + u_4^4\}$$

much simpler…

Groebner basis in u_1, u_2, u_3, u_4 and auxiliary variables s_1, s_2, s_3, s_4. s_i equals the degree-i symmetric polynomial of u's.

An efficient Groebner basis algorithm for symmetric ideals is needed

will try “symodstd_lib” in Singular soon

Fast Groebner basis computation needed …

comparing with Faugere’s FGb package, over Z/9001, on a laptop

	Slimgb	FGb
L=7, S=3	~0 secs	~0 secs
L=8, S=4	~5 secs	~0 secs
L=9, S=4	~183 secs	~5 secs
L=10, S=5	? >8h	~209 secs
L=12, S=5	? >8h	~835 secs
L=12, S=6	? >8h	crashed

sba, mathicgb to be tested …

Summary

- Algebraic geometry approach for theoretical physics frontier problems

  scattering amplitudes

  integrability

- Singular is very powerful, and important for our research
- more efficient intersect, syz, slimgb and symmetric ideal tools are very useful