Amplitudes – Tutorial 1

We consider the special unitary group $SU(N)$ with the hermitian matrices T^a as generators of the fundamental representation, $a \in \{1, \dots, N^2 - 1\}$, and fix the conventions

$$[T^a, T^b] = i f^{abc} T^c, \qquad \mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}\,. \tag{1}$$

Fierz identity: Prove the Fierz identity

$$T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right)\,. \tag{2}$$

Traces: Using the Fierz identity show that

$$\begin{aligned} \mathrm{Tr}(T^a X)\mathrm{Tr}(T^a Y) &= \frac{1}{2}\left(\mathrm{Tr}(XY) - \frac{1}{N}\mathrm{Tr}(X)\mathrm{Tr}(Y)\right) \\ \mathrm{Tr}(T^a X T^a Y) &= \frac{1}{2}\left(\mathrm{Tr}(X)\mathrm{Tr}(Y) - \frac{1}{N}\mathrm{Tr}(XY)\right)\,, \end{aligned} \tag{3}$$

and compute

$$\mathrm{Tr}(T^a T^b T^b T^a)\,, \qquad \mathrm{Tr}(T^a T^b T^a T^b)\,. \tag{4}$$

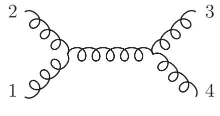
Adjoint representation: The generators T^a satisfy the Jacobi identity

$$[[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0\,. \tag{5}$$

Write the Jacobi identity in terms of the structure constants f^{abc} and show that these constitute a valid representation $\mathcal{T}^a$ for the group $SU(N)$, i.e., they satisfy

$$[\mathcal{T}^a, \mathcal{T}^b] = i f^{abc}\mathcal{T}^c\,. \tag{6}$$

Colour factor of gluonic amplitudes: Consider the contribution



to the 4-point tree-level gluonic amplitude. We will neglect all kinematic information and focus on the colour structure. Each three-gluon vertex contributes a factor of if^{abc}, and it is convenient to rewrite them in terms of traces of the generators T^a. Show that

$$i f^{abc} = 2\left(\mathrm{Tr}(T^a T^b T^c) - \mathrm{Tr}(T^b T^a T^c)\right)\,, \tag{7}$$

and write the colour factor of the diagram above in terms of traces of the T^{a_i}, $i = 1, \dots, 4$ using the trace identities computed above.