**Exercises to General Relativity II** SS 2014

**Exercise 1** *Charter-Penrose Diagram for Minkowski space*
Consider four-dimensional Minkowski space in polar coordinates,

$$ds^2 = -dt^2 + dr^2 + r^2\, d\Omega_2^2\,, \tag{1}$$
$$d\Omega_2^2 = (d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{2}$$

a) Show that the coordinate change, $(r+t) = \tan(R+T)$ and $(r-t) = \tan(R-T)$ gives the metric,

$$ds^2 = f(T,R)\,(-dT^2 + dR^2 + \sin^2(R)\,d\Omega_2^2)\,, \tag{3}$$

and determine the function $f(T,R)$.

b) Give the coordinate ranges for the time coordinate T and the radial coordinate R for which Minkowski space is covered once in the new coordinates.

c) Draw the Carter-Penrose diagram and identify $i^\pm$, $i^0$, $\mathcal{I}^\pm$.

d) Give the paramtrization of the following geodesics through the origin $R = 0$:

- light rays
- two distinct time-like geodesics
- two distinct space-like geodesics

Draw the geodesics qualitatively in the Penrose diagram.

**Exercise 2** *Maxwell-Einstein equations*
The aim of this exercise is to derive the solution of the Einstein-Maxwell equations of a static point-like charge. Symmetry considerations allow to consider the following ansatz for metric and electromagnetic potential,

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\Omega_2^2\,,$$
$$A_t = \phi(r)\,,\quad A_r = A_\theta = A_\phi = 0\,. \tag{4}$$

a) Show that the current of a point-like charge $Q$ at the origin of spherical coordinates is given by $j^\mu(r) = \{Q\delta(r), 0, 0, 0\}$. (Here we use the definition of the $\delta(r)$ distribution to be $\int d\theta d\phi dr\, r^2 \sin(\theta)\delta(r) f(\vec{x}) = f(\vec{0})$.)

b) Show that the inhomogenous Maxwell equations for a generic metric tensor are given by,

$$\nabla_\mu F^{\mu\nu} = -4\pi j^\nu \quad \rightarrow \quad \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}F^{\mu\nu}) = -4\pi j^\nu\,. \tag{5}$$

c) Solve the Maxwell equations for the ansatz (4) for $r > 0$ up to the integration constant. Later fix the integration constant by comparing the volume integrals of the current,

$$\int_{R^3,t=t_0} (\frac{1}{3!}\sqrt{g}\varepsilon_{\nu\mu_1\mu_2\mu_3}dx^{\mu_1}dx^{\mu_2}dx^{\mu_3})\, j^\nu \,, \tag{6}$$

and the left-hand side of the field equations.

**Discussion 1** *Reissner Nordstrøm metric*

The Reissner Nordstrøm metric solves the Einstein-Maxwell equations. To show this one starts from a spherically symmetric ansatz and fixes the remaining functions using the Einstein-Maxwell equations.

The ansatz for the metric and the electromagnetic potential is given by,

$$ds^2 = -A(r)\, dt^2 + B(r)\, dr^2 + r^2\, d\Omega_2^2 \,, \tag{7}$$

$$A_t = \phi(r)\,, \quad A_r = A_\theta = A_\phi = 0\,. \tag{8}$$

The Einstein equations are given by,

$$R_{\mu\nu} = 8\pi G_N\,(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\rho_\rho)\,. \tag{9}$$

The energy-momentum tensor and the non-vanishing components of the Ricci tensor are given by,

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)\,, \tag{10}$$

$$R_{tt} = \frac{A''}{2B} - \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{rB}\,, \tag{11}$$

$$R_{rr} = -\frac{A''}{2A} + \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{B'}{rB}\,, \tag{12}$$

$$R_{\theta\theta} = 1 - \frac{1}{B} - \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right)\,, \tag{13}$$

$$R_{\phi\phi} = \sin^2(\theta) R_{\theta\theta}\,. \tag{14}$$

The aim of this exercise will be to find the solutions of the field equations,

$$A = B^{-1} = 1 - \frac{m}{r} + \frac{q^2}{r^2}\,, \quad \phi = -\frac{Q}{r}\,, \quad q^2 = G_N Q^2\,. \tag{15}$$

a) Compute the field strength tensor $F_{\mu\nu}$.

b) Compute the energy-momentum tensor of the electromagnetic field.

c) Show that the energy-momentum tensor is trace less $T^\mu_\mu = 0$.

d) Use the explicit form of $T_{\mu\nu}$ eqns. (11) and (12) to show that $BR_{tt} + AR_{rr} = 0$ and that this implies $A(r) = B(r)^{-1} =: f(r)$.

e) The solutions of the inhomogenous Maxwell equations is given by $\phi(r) = -Q/r$. Use this in combination with the $R_{\theta\theta}$ Einstein equations to show that $f(r)$ is given by,

$$f(r) = 1 - \frac{m}{r} + \frac{q^2}{r^2}\,. \tag{16}$$