Exercise 3 *Kruskal Szekeres coordinates.*
Given is the four-dimensional metric,

$$ds^2 = \frac{32m^3}{r} e^{-r/2m}(-dT^2 + dX^2) + r(X,T)^2\, d\Omega_2^2\,, \tag{1}$$

$$r(T,X) \quad \text{from} \quad X^2 - T^2 = \left(\frac{r}{2m} - 1\right) e^{r/2m}\,. \tag{2}$$

a) Show that the coordinate transformation of the above metric give the Schwarzschild metric in each of the four regions a,b,c and d,

$$\text{regions a/b:} \quad r > 2m\,, \tag{3}$$

$$T = \left(\frac{r}{2m} - 1\right)^{1/2} e^{r/4m} \sinh(t/4m)\,, \tag{4}$$

$$X = \left(\frac{r}{2m} - 1\right)^{1/2} e^{r/4m} \cosh(t/4m) \cdot (\pm 1)\,, \tag{5}$$

$$\text{regions c/d:} \quad 0 < r < 2m\,, \tag{6}$$

$$X = \left(1 - \frac{r}{2m}\right)^{1/2} e^{r/4m} \sinh(t/4m)\,, \tag{7}$$

$$T = \left(1 - \frac{r}{2m}\right)^{1/2} e^{r/4m} \cosh(t/4m) \cdot (\pm 1)\,. \tag{8}$$

$$\tag{9}$$

b) Draw the r and t coordinate lines in (X,T) coordinate space and show that two copies of the (r,t) coordinate patch (with $r > 0$ and $t \in R$) have to be used to cover (X,T)-coordinates once.

c) Arrange the regions a,b,c and d in order to obtain a the geodesically complete space-time that solves the vacuum Einstein equations for $r > 0$.

Exercise 4 *Basic properties of sigma models.*
Consider the σ-model,

$$S = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau\, \sqrt{g} g^{mn}(\sigma,\tau)\partial_m X^\nu(\sigma,\tau)\partial_n X^\nu(\sigma,\tau)\, G_{\mu\nu}(X)\,. \tag{10}$$

a) Derive the field equation of the world-sheet metric g_{mn}.

b) Assume a flat embedding space $G_{\mu\nu}(X) = \eta_{\mu\nu}$ and derive the conserved currents and charges associated to translations in target space,

$$X^\mu(\sigma,\tau) \to X^\mu(\sigma,\tau) + \varepsilon^\mu\,. \tag{11}$$

Assume that the space coordinate σ is periodic; $\sigma \in [0, 2\pi)$ and $X^\mu(\sigma,\tau) = X^\mu(\sigma + 2\pi,\tau)$, $g^{mn}(\sigma,\tau) = g^{mn}(\sigma + 2\pi,\tau)$.

c) Assume a flat embedding space $G_{\mu\nu}(X) = \eta_{\mu\nu}$ and derive the field equations for the fields $X^\mu(\sigma, \tau)$. Verify that the charges are in fact conserved.

Discussion 3 *Field equations for a scalar field.*
Consider a scalar field ϕ in the Scharzschild geometry,

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2\, d\Omega_2^2\,, \quad f(r) = \left(1 - \frac{2m}{r}\right)\,, \tag{12}$$

with

$$S = -\frac{1}{2}\int dx^4 \sqrt{g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)\,. \tag{13}$$

a) Give the field equations for the scalar field $\phi(t, r, \theta, \varphi)$.

b) Work out the field equation for $h_{l,m}(t, r)$ for the ansatz,

$$\phi(t, r, \theta, \varphi) = h_{lm}(t, r)\, Y_{lm}(\theta, \varphi)\,, \tag{14}$$

using,

$$\frac{1}{\sin(\theta)}\partial_\theta\left(\sin(\theta)\partial_\theta Y_{lm}(\theta,\varphi)\right) + \frac{1}{\sin^2(\theta)}\partial_\varphi^2 Y_{lm}(\theta,\varphi) = -l(l+1)Y_{lm}(\theta,\varphi)\,.$$

c) Transform the differential equation to tortoise coordinates with $\partial_r = dr^*/dr\partial_{r^*}$ and $dr^*/dr = f^{-1}$.

d) Show that the redefinition of the fields $h_{lm}(t, r^*) = \psi_{lm}(t, r^*)/r$ leads to the field equation,

$$(\partial_t^2 - \partial_{r^*}^2)\psi_{lm}(t, r^*) + V_l(r^*)\psi_{lm}(t, r^*)\,, \quad V_l(r^*) = f(r)\left(\frac{l(l+1)}{r^2} + \frac{2m}{r^3}\right)\,. \tag{15}$$

e) Discuss the solutions to the above field equation in the asymtotic regions near the horizon and near spatial infinity.

Discussion 4 *Near horizon geometry of Black Holes.*
Consider the Schwarzschild metric and the Reissner Nordstrøm metric,

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2\, d\Omega_2^2\,, \tag{16}$$

for which $f(r)$ takes the form,

$$f(r) = 1 - \frac{2m_S}{r}\,, \quad \text{or} \quad f(r) = 1 - \frac{2m_{RN}}{r} + \frac{q^2}{r^2}\,, \tag{17}$$

respectively.

a) Identify the locations of the event horizon(s). (Consider only the parameter ranges with $m_S > 0$ and $m_{RN} > 0$.)

b) Consider the hypersurfaces of the singularities $r \to 0$. Are the singularities spacelike or timelike?

c) For which cases can we speak of naked singularities?

d) Consider the near-horizon geometries; which of the two cases can be found when we expand the metric close to the location of the apparent horizon,

$$\text{Rindler-like:} \qquad ds^2 = -\kappa^2 \rho^2 dt^2 + d\rho^2 + R^2 \, d\Omega_2^2 \,, \tag{18}$$

$$AdS_2 \times S_2 : \qquad ds^2 = R^2 \frac{(-dt^2 + dy^2)}{y^2} + R^2 \, d\Omega_2^2 \,, \tag{19}$$

and determine the parameters κ and R.