Exercise 5 *Virasoro algebra*

In the following two representations of the world-sheet generators of the conformal group will be compared.

a) Consider the world-sheet vector fields $\xi_n = i\, e^{in\sigma^+}\partial_+$ and show that they fulfill the classical Virasoro algebra,

$$[\xi_n, \xi_m] = (n-m)\xi_{m+n}\,. \tag{1}$$

b) Consider next the Virasoro generators,

$$L_n = -\frac{1}{2}\sum_{m=-\infty}^{\infty} \eta_{\mu\nu}\, \alpha^\mu_m \alpha^\nu_{n-m}\,. \tag{2}$$

Use the commutator relations,

$$[\alpha^\mu_n, \alpha^\nu_m] = -m\delta_{m,-n}\eta^{\mu\nu}\,, \quad [p^\mu, x^\nu] = -i\eta^{\mu\nu}\,, \quad [\tilde{\alpha}_m, \tilde{\alpha}_n] = 0\,, \tag{3}$$

to show that the Virasoro generators act in to following way on the mode operators α^ρ_l,

$$[L_n, \alpha^\rho_l] = l\, \alpha^\rho_{l+n}\,. \tag{4}$$

c) Use the commutator relation (4) to show that the operators L_n do in fact fulfill the Virasoro algebra,

$$[L_n, L_m] = (m-n)L_{m+n}\,. \tag{5}$$

d) Consider the mode expansion of the operators $X^\mu(\tau,\sigma)$,

$$X^\mu(\tau,\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)\,, \tag{6}$$

$$X^\mu_L(\sigma^+) = \frac{1}{2}x^\mu_L + \frac{\alpha^\mu_0}{\sqrt{4\pi T}}\sigma^+ + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n\, e^{-in\sigma^+}\,, \tag{7}$$

$$X^\mu_R(\sigma^-) = \frac{1}{2}x^\mu_R + \frac{\tilde{\alpha}^\mu_0}{\sqrt{4\pi T}}\sigma^- + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n i\, e^{-in\sigma^-}\,, \tag{8}$$

with $\sigma^\pm = (\tau \pm \sigma)$ and compare the action of the vector fields ξ_n one the operators $X^\mu(\tau,\sigma)$ (acting as derivatives $\xi_n(X^\mu(\tau,\sigma))$) with the commutators $[L_n, X^\mu(\tau,\sigma)]$.

Exercise 6 *Classical string in light-cone gauge*

Consider the light-cone coordinates in target space,

$$x^+ = x^0 + x^1 \,, \quad x^- = x^0 - x^1 \,, \tag{9}$$

$$ds^2 = -\eta_{+-} dx^+ dx^- + \eta_{-+} dx^+ dx^- - dx^i dx^i \,, \quad \eta_{+-} = \eta_{-+} = -\frac{1}{2} \quad \eta 6{+-} = \eta^{-+} = -2 \tag{10}$$

such that the embedding functions $X^\mu(\tau, \sigma)$ are written in the form,

$$X^-(\tau, \sigma) = X^0(\tau, \sigma) - X^1(\tau, \sigma) \,, \tag{11}$$

$$X^+(\tau, \sigma) = X^0(\tau, \sigma) + X^1(\tau, \sigma) \,, \tag{12}$$

and $X^i(\tau, \sigma)$ with $i = \{2, \ldots, D\}$. Furthermore, the reparametrization invariance on the world sheet can be used to relate the world sheet time to the light cone coordinate X^+, $\tau \to \frac{1}{2}(f^+(\sigma^+) + f^-(\sigma^-))$, so that the simple form $X^+(\tau, \sigma) = (\alpha_0^+ + \tilde{\alpha}_0^+)/\sqrt{4\pi T}\tau$ can be obtained.

The mode expansion of the remaining fields $\mu = \{-, 2, ..., D\}$ is given in the standard from,

$$X^\mu(\tau, \sigma) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-) \,, \tag{13}$$

$$X_L^\mu(\sigma^+) = \frac{1}{2} x_L^\mu + \frac{\alpha_0^\mu}{\sqrt{4\pi T}} \sigma^+ + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu \, e^{-in\sigma^+} \,, \tag{14}$$

$$X_R^\mu(\sigma^-) = \frac{1}{2} x_R^\mu + \frac{\tilde{\alpha}_0^\mu}{\sqrt{4\pi T}} \sigma^- + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu i \, e^{-in\sigma^-} \,, \tag{15}$$

with $\sigma^\pm = \tau \pm \sigma$. In these coordinates the Virasoro constraints take a particularly simple form allowing to obtain explicit solutions for the classical string equations. The following steps should be a guide through the construction.

a) Compute the total momentum of the string,

$$p^\mu = T \int d\sigma \dot{X}^\mu \,. \tag{16}$$

and express the modes $\alpha_0^\mu, \tilde{\alpha}_0^\mu$ in terms of the total momenta $p^\mu = \{p^+, p^-, p^2, \ldots, p^D\}$.

b) Compute the Virasoro generators,

$$L_n := T \int_0^{2\pi} d\sigma \, T_{++} e^{in\sigma^+} \,, \quad \tilde{L}_n := T \int_0^{2\pi} d\sigma \, T_{--} e^{in\sigma^-} \tag{17}$$

in terms of the modes $\alpha_m^\mu, \tilde{\alpha}_m^\mu$.

c) Solve the Virasoro constraints ($L_n = 0 = \tilde{L}_n$) under the assumption that only one frequency is excited in the 2-3 plane; $\{\alpha_n^2, \alpha_n^3, \tilde{\alpha}_n^2, \tilde{\alpha}_n^3\} \neq 0$. It is best to consider this case in the center-of-mass frame system with $x^\mu = p^i = 0$.

d) The angular momentum of a closed string is given by,

$$J_{\mu\nu} = T \int d\sigma \left(X_\mu \dot{X}_\nu - X_\nu \dot{X}_\mu \right) . \tag{18}$$

Compute the mass $m^2 = p^2$ and the angular momentum J^{23} of the string solution of the previous task. Plot the Regge trajectory $r(n) := J^{23}/m^2$ as a function of the mode number n. What is the slope of the Regge trajectory?

Discussion 5 *Komar integral*

The aim of this section is to compute conserved quantities of blackhole geometries with Killing vectors.

To this end consider a geometry with a Killing vector field K_ν with $\nabla_{(\mu}K_{\nu)} := (\nabla_\mu K_\nu + \nabla_\nu K_\mu)/2 = 0$. In the following we will denote the Ricci tensor with $R_{\mu\nu}$ and the Ricci scalar with $R = R_{\mu\nu}g^{\mu\nu}$.

a) Show that the current J^μ,

$$J^\mu = K_\nu R^{\mu\nu} \tag{19}$$

is conserved $\nabla_\mu J^\mu = 0$. Use the relation $\nabla_\mu R^{\mu\nu} = \frac{1}{2}\nabla^\nu R$.

b) Show that the following relation holds for an anti-symmetric tensor $K^{\mu_1\mu_2\ldots\mu_{m+1}}$,

$$\sqrt{g}\varepsilon_{\mu_1\ldots\mu_m\ldots\mu_D}\nabla_\rho K^{\rho\mu_1\mu_2\ldots\mu_m} = \partial_\rho \left(\sqrt{g}\varepsilon_{\mu_1\ldots\mu_m\ldots\mu_D} K^{\rho\mu_1\mu_2\ldots\mu_m} \right) , \tag{20}$$

and review Stoke's theorem,

$$\int_\Sigma \sqrt{g}\varepsilon_{\mu_1\ldots\mu_m\ldots\mu_D}\nabla_\rho K^{\rho\mu_1\mu_2\ldots\mu_m} \frac{dx^{\mu_{m+1}}\cdots dx^{\mu_D}}{(D-m)!} \tag{21}$$

$$= (-1)^m \int_{\partial\Sigma} \sqrt{g}\varepsilon_{\mu_1\ldots\mu_m\ldots\mu_D} K^{\mu_1\mu_2\ldots\mu_m\mu_{m+1}} \frac{dx^{\mu_{m+2}}\cdots dx^{\mu_D}}{(D-m-1)!} , \tag{22}$$

where the notion $\partial\Sigma$ stands for the boundary of the surface Σ.

c) The conserved charge is then given by the following integral of the current over a space-like slice of the space time,

$$Q_K = \frac{1}{4\pi G_N} \int_\Sigma \sqrt{g}\varepsilon_{\mu\nu\rho\sigma} J^\mu \frac{dx^\nu dx^\rho dx^\sigma}{3!} . \tag{23}$$

First use the relation $\nabla_\mu \nabla_\nu K^\mu = K^\mu R_{\mu\nu}$ and then Stoke's theorem (based on equation 20) to rewite the above integral into the Komar integral,

$$Q_K = \frac{1}{4\pi G_N} \int_{\partial\Sigma} \sqrt{g}\varepsilon_{\mu\nu\rho\sigma} \nabla^\mu K^\nu \frac{dx^\rho dx^\sigma}{2!} . \tag{24}$$

The later integral allows to measure charges (mass, spin) of a blackhole geometries in the asymtotic region avoiding the event horizon as well as the curvature singularity.