Exercises to General Relativity II SS 2014

Exercise 7 *BRST charge*

The aim of this exercise is to obtain the form of the BRST current for the bosonic string. To this end consider the gauge fixed action of the bosonic string in light cone coordinates,

$$\mathcal{L} = \partial_+ X^\mu \partial_- X_\mu + (b_{++}\partial_- c^+ + b_{--}\partial_+ c^-) \,. \tag{1}$$

The BRST transformations are given by,

$$\begin{aligned} sX^\mu &= c^+\partial_+ X^\mu + c^-\partial_- X^\mu \,, &(2)\\ sb_{++} &= T_{++} \,, &(3)\\ sb_{--} &= T_{--} \,, &(4)\\ sc^+ &= c^+\partial_+ c^+ + c^-\partial_- c^+ \,, &(5)\\ sc^- &= c^+\partial_+ c^- + c^-\partial_- c^- \,, &(6) \end{aligned}$$

where the components of the energy-momentum tensor T_{++} are given by,

$$\begin{aligned} T_{++} &= T^X_{++} + T^{gh}_{++} \,, &(7)\\ T^X_{++} &= \partial_+ X^\mu \partial_+ X_\mu \,, \quad T^{gh}_{++} = 2b_{++}\partial_+ c^+ + \partial_+ b_{++} c^+ \,, &(8) \end{aligned}$$

and analogously for T_{--}.

a) Compute the equations of motion for the fields X, c^+, c^-, b_{--} and b_{++}.

b) Show that the above Lagrangian transforms into a total derivative modulo equations of motion under the above BRST variations,

$$s\mathcal{L} = \partial_-(c^-\partial_+ X^\mu \partial_- X_\mu) + \partial_+(c^+\partial_+ X^\mu \partial_- X_\mu) + e.o.m. \tag{9}$$

c) The formula for the Noether current associated to the BRST symmetry is given by,

$$J^m_{BRST} = -sX^\mu \frac{\partial \mathcal{L}}{\partial \partial_m X^\mu} - sc^k \frac{\partial \mathcal{L}}{\partial \partial_m c^k} + c^m(\partial_+ X^\mu \partial_- X_\mu) \,. \tag{10}$$

Compute the Noether current explicitly. (It is OK to use the equations of motion in this computation.) Show that the components of the currents J^m_{BRST} can be written as,

$$J^-_{BRST} \sim -c^+(T^{(X)}_{++} + \frac{1}{2}T^{gh}_{++}) \,, \quad J^+_{BRST} \sim -c^-(T^{(X)}_{--} + \frac{1}{2}T^{gh}_{--}) \,. \tag{11}$$

Exercise 8 *Zero mode Fock space*

The canonical quantizaton of fermionic fields (ghosts as well as regular fermions) on the world sheet leads to anti-commutator relations for the mode operators of the form,

$$\{\psi^l_r, \psi^m_s\} = \delta^{lm}\delta_{r+s,0}\,. \tag{12}$$

with the modes related to the fermionic fields as

$$\psi^l_+(\sigma,\tau) = \sqrt{\frac{\alpha'}{2}}\sum_{r\in Z+\varphi} \psi^l_r\, e^{-ir\sigma^+}\,. \tag{13}$$

Here r and s are the mode labels and the indices l and m label fermionic fields species, $l = 1, N$. For the present exercise we can assume that $\varphi = 0$.

The aim of this exercise will be to construct the Fock space for the zero-mode sector generated by the modes with $r = s = 0$. Here it turns out that the anti-commutator relations are identical to the one of a Clifford algebra in N dimensions. Thus the Fock space for these operators can be identified with representation space of Clifford algebras and thus Dirac spinors in N dimensions. (In the following consider N to be an even number.)

a) What is the dimensionality of Dirac spinors in N dimensions.

b) Form linear combinations of the mode operators $\alpha^k = (\psi^{2k}_0 + i\psi^{2k+1}_0)$ and $\alpha^{k\dagger} = (\psi^{2k}_0 - i\psi^{2k+1}_0)$. Give the (anti) commutator relations for these operators and show that they can be interpreted as fermionic creation and annihilation operators, respectively

c) Count the states in the Fock space created by the fermiononic creation operators and compare to the expected dimensionality of Dirac spinors.

Remark: Some details about spinor represenations can be found in Polchnisky's String Theory book, Volume II, Appendix B.

Discussion 6 *World sheet fermions*

Consider the world-sheet Lagrangian including fermions in world-sheet light-cone coordinates,

$$\mathcal{L} \sim (\partial_+ X \partial_- X + i\psi_+ \partial_- \psi_+ + i\psi_- \partial_+ \psi_-) \,. \tag{14}$$

The fields $\psi_\pm$ denote the one-dimensional Weyl spinors in two dimensions. Conformal transformations parametrized by the conformal Killing vectors $\{v^+(\sigma^+), v^-(\sigma^-)\}$ are given by

$$\delta X = v^+ \partial_+ X + v^- \partial_- X \,, \tag{15}$$

$$\delta \psi_+ = v^+ \partial_+ \psi_+ + h_\psi \partial_+ v^+ \psi_+ \,, \tag{16}$$

$$\delta \psi_- = v^- \partial_- \psi_- + \bar{h}_\psi \partial_- v^- \psi_- \,. \tag{17}$$

The real constants h_ψ and $\bar{h}_\psi$ denote the conformal weights of the fields. Their value will be obtained below.

a) Show that the Lagrangian transforms into a total derivative under the conformal transformations,

$$\delta \mathcal{L} = \partial_+ \mathcal{J}^+ + \partial_- \mathcal{J}^- \tag{18}$$

if the conformal weights are $h_\psi = \bar{h}_\psi = 1/2$. Use this information to compute the Noether currents of the above symmetry transformations. These currents j_m allow to obtain the energy-momentum tensors of the theory, $j_+ = v^+ T_{++}$ and $j_- = v^- T_{--}$.

b) Compute the equations of motion for the fields $\psi_\pm$ and X, and argue that the following mode expansions solve the field equations for the fermions,

$$\psi_+(\sigma, \tau) = \sum_{r \in Z + \varphi} \psi_r \, e^{-ir\sigma^+} \,, \tag{19}$$

$$\psi_-(\sigma, \tau) = \sum_{r \in Z + \varphi} \tilde{\psi}_r \, e^{-ir\sigma^-} \,. \tag{20}$$

with an unspecified phase shift ϕ.

c) Insert the mode expansion into the energy-momentum tensors and compute the Virasoro generators for fermion theory. Fix the unspecified phase shift φ in order to obtain a periodic energy-momentum tensor.