Exercise 9 *Normal-ordering constants.*

Consider free conformal field theories in two dimension. In the quantised Virasoro algebra normal ordering constants appear which can be computed by regularising and renormalising the theory. Starting point are the (anti-) symmetrised Virasoro generators,

$$L_n^{(X)} = -\frac{1}{2}\sum_m (\alpha_{n-m}\alpha_m + \alpha_m\alpha_{n-m})/2\,, \tag{1}$$

$$L_n^{(c)} = \sum_m (n+m)(b_{n-m}c_m - c_m b_{n-m})/2\,, \tag{2}$$

$$L_n^{(\psi)} = -\frac{1}{2}\sum_r \left(\frac{n}{2} - r\right)(\psi_{n-r}\psi_r - \psi_r\psi_{n-r})/2\,, \tag{3}$$

$$L_n^{(\gamma)} = \sum_r \left(\frac{n}{2} + r\right)(\beta_{n-r}\gamma_r + \gamma_r\beta_{n-r})/2\,. \tag{4}$$

of the scalars, the b-c-system, fermions and the β-γ-system. (For simplicity it is sufficient to consider the left moving sector.) Here the labels m run over integer values from minus infinity to plus infinity. The labels r may run over integer values (NS-sector) or half integer values (R-sector). Note, that the modes β_m and γ_m are bosonic, while the modes c_m and b_m are fermionic.

The aim of this exercise is to compute the normal-ordering constants $a^{(\Phi)}$ by rewriting the Virasoro generators L_0 into normal ordered form,

$$L_n^{(X)} = -\frac{1}{2}\sum_m :\alpha_{n-m}\alpha_m: + a^{(X)}\delta_{n,0}\,, \qquad L_n^{(c)} = \sum_m (n+m) :b_{n-m}c_m: + a^{(c)}\delta_{n,0}\,,$$

$$L_n^{(\psi)} = -\frac{1}{2}\sum_r (\frac{n}{2} - r) :\psi_{n-r}\psi_r: + a^{(\psi)}\delta_{n,0}\,, \qquad L_n^{(\gamma)} = \sum_r (\frac{n}{2} + r) :\beta_{n-r}\gamma_r: + a^{(\gamma)}\delta_{n,0}\,.$$

with

$$[\alpha_m^\mu, \alpha_n^\nu] = -m\,\eta^{\mu\nu}\delta_{m+n,0}\,, \qquad \{\psi_m^\mu, \psi_n^\nu\} = \eta^{\mu\nu}\delta_{m+n,0}\,, \tag{5}$$

$$[\gamma_r, \beta_s] = \delta_{r+s,0}\,, \qquad \{c_m, b_n\} = \delta_{m+n,0}\,. \tag{6}$$

Follow the following two approaches to obtain the values of the constants $a^{(\Phi)}$.

a) The infinite sums can be interpreted by regularizing the theories,

$$\lim_{\alpha\to 0}\sum f^{(\Phi)}(n)e^{-\alpha n} \to \frac{a^{(\Phi)}_{\text{div}}}{\alpha^2} + a^{(\Phi)} + \mathcal{O}(\alpha)\,. \tag{7}$$

and subtracting the divergent parts to obtain $a^{(\Phi)}$.

b) Use the rule,

$$\sum_{m=1}^{\infty} m \to -\frac{1}{12}\,. \tag{8}$$

to rewrite the infinite sums.

The results are $a^{(X)} = -D/24$, $a^{(c)} = 1/12$ for the bosonic sector. They are $a^{(\psi)}_{1/2} = -D/48$, $a^{(\gamma)} = 1/24$ for half integer modes (NS-sector) of ψ and the $\beta - \gamma$-system. Finally, $a^{(\psi)} = D/24$, $a^{(\gamma)} = -1/12$ for integer modes.

Exercise 10 *Towards the critical dimension of the bosonic string.*
The normal ordering constants of the matter and ghost fields deform the Virasoro algebra through constant shifts,

$$A_m = \frac{A_2 - 2A_1}{6}\, m^3 - \frac{A_2 - 8A_1}{6}\, m\,. \tag{9}$$

which enter the Virasoro algebra like,

$$[L_m, L_n] = (m-n)L_{m+n} + A_m \delta_{m+n}\,. \tag{10}$$

Compute the constants A_1 and A_2 from the expectation values,

$$A_m = \langle \uparrow | L_m L_{-m} - 2mL_0 | \downarrow \rangle\,. \tag{11}$$

For simplicity insert the normal ordered Virasoro generators for the bosonic sector, $L_m = L_m^{(X)} + L_m^{(c)} + a^{(X,c)}\delta_{m,0}$. Treat the normal ordering constant $a^{(X,c)}$ as an unknown in this exercise.