Exercise 34 *Cosmologies.*

Our cosmology can be modeled by a space time with the metric

$$ds^2 = -dt^2 + R(t)^2 \left(d\mathbf{x} \cdot d\mathbf{x} + \frac{k\,(\mathbf{x} \cdot d\mathbf{x})^2}{(1 - k\,\mathbf{x}^2)} \right) , \quad \mathbf{x} = \{x^1, x^2, x^3\} . \tag{1}$$

The dynamics of matter on the space time determines the discrete parameter k ($k = -1, 0, 1$) as well as the time dependent function $R(t)$. However, here we will be concerned with those properties of the models which are independent of these parameters.

a) In this model galaxies are described as point-like objects moving along geodesics. Show that a valid trajectory for galaxies is to remain at rest at a fixed position in $\mathbf{x}$-space, $\mathbf{x}(t) = \mathbf{x}_0$. To this end consider the geodesic equations and argue which Christoffel symbol(s) have to vanish for this to be true. Show by explicit computation that the respective Christoffel symbol(s) in fact vanish.

b) Convert the metric to polar coordinates $\{r, \theta, \phi\}$ using $x^1 = r \sin\theta \cos\phi$, $x^2 = r \sin\theta \sin\phi$ and $x^3 = r\cos\theta$.

Exercise 35 *Tensor Transformation Properties.*

Consider the following tensors written in terms of the coordinate basis vectors and one-forms associated to the coordinate chart $\{x^i\}$,

$$\alpha = \alpha_i dx^i , \quad \beta = \beta_{ij} dx^i dx^j , \quad v = v^i \frac{\partial}{\partial x^i} . \tag{2}$$

Express the basis vectors and one-forms in terms of the coordinate basis associated to x'^k with $x'^k(x)$ and read off the transformation behaviour of the components of the tensors.

Consider next the covariant derivatives of the components of the above tensors in component form,

$$T_{ij} := D_i \alpha_j = \partial_i \alpha_j - \Gamma^k_{ij} \alpha_k , \tag{3}$$

$$T_i^j := D_i v^j = \partial_i v^j + \Gamma^j_{ik} v^k , \tag{4}$$

$$T_{ijk} := D_i \beta_{jk} = \partial_i \beta_{jk} - \Gamma^l_{ij} \beta_{lk} - \Gamma^l_{ik} \beta_{jl} . \tag{5}$$

Show that the above covariant derivatives transfrom like tensors under a coordinate change form $\{x^i\}$ to $\{x'^i\}$. To this end use the transformation properties of the connection coefficients,

$$(\Gamma')^l_{mn} = \frac{\partial x'^l}{\partial x^k} \frac{\partial^2 x^k}{\partial x'^m \partial x'^n} + \frac{\partial x'^l}{\partial x^k} \frac{\partial x^i}{\partial x'^m} \frac{\partial x^j}{\partial x'^n} \Gamma^k_{ij} . \tag{6}$$

Exercise 36 *Parallel transport.*

Consider the two sphere with the metric, $ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$. Show that the space is approximated by the Eucliden metric of flat space close to the poles. What is the interpretation of the angle θ close to the poles? Furthermore, show that close to the equator ($\theta = \pi/2$) the metric resembles the one of flat space with periodic boundary conditions.

The only non-vanishing Christoffel symbols of the two sphere are $\Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \cot\theta$ and $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$. Give the Christoffel symbols close to equator and the poles.

A family of vectors $v^i(x(s))\partial_i$ is called parallel along a given curve $x^j(s)$ if it obeys the differential equation,

$$\frac{dx^i(s)}{ds}\left(\partial_i v^j(x(s)) + \Gamma^j_{ik} v^k(x(s))\right) = 0\,. \tag{7}$$

Solve the differential equation of the given two-dimensional space for the curves $x^\mu(\phi) = \{\phi, \theta_0\}$ with $\phi \in (0, 2\pi]$ and $\theta_0 =$const. for the components $v^i(x(\phi))$ as a function of ϕ. Discuss the solutions as the parameter θ_0 is varied. In particular discuss the cases with $\theta_0 = 0$, $\theta_0 = \pi/2$ and compare to the flat-space intuition of the parallel transport.