Exercise 37 *Tensors.*

The addition rule for two tensors S and T of the same rank (k, l) is defined by,

$$(S+T)(\omega_1, ..., \omega_k, v_1, ..., v_l) := S(\omega_1, ..., \omega_k, v_1, ..., v_l) + T(\omega_1, ..., \omega_k, v_1, ..., v_l)\,. \qquad (1)$$

Show, that the sum of the tensors $(S+T)$ is in fact a tensor being a multi linear map in all its arguments.

The tensor product of a tensor T with rank (k', l') and a tensor S with rank $(k-k', l-l')$ (with $k \geq k'$ and $l \geq l'$) is defined by,

$$(S \otimes T)(\omega_1, ..., \omega_k, v_1, ..., v_l) := S(\omega_1, ..., \omega_{k'}, v_1, ..., v_{l'}) \cdot T(\omega_{k'+1}, ..., \omega_k, v_{l'+1}, ..., v_l)\,. \qquad (2)$$

Show that this definition of the product gives a tensor of rank (k, l).

Furthermore, show that the tensor product is associative such that, $S \otimes (T \otimes U) = (S \otimes T) \otimes U$. For simplicity show this assuming that S, T and U are rank $(1, 0)$ tensors.

Next start from the (k, l) tensor T,

$$T = T^{i_1 \dots i_k}_{j_1 \dots j_l}\ \partial_{i_1} \otimes ...\partial_{i_k} \otimes dx^{j_1} \otimes ... \otimes dx^{j_l}\,, \qquad (3)$$

and change to a new coordinate basis for geneneral coordinate transformations $y^i(x)_{i=1,D}$ with the new basis vectors $\frac{\partial}{\partial y^j}$ and one-forms dy^j. How do the components $T^{i_1 i_2 \dots i_k}_{j_1 j_2 \dots j_l}$ of the tensor transform?

Exercise 37 *Binachi Identity.*

Show/argue that the Jacobi identity,

$$[D_i, [D_j, D_k]] + [D_j, [D_k, D_i]] + [D_k, [D_i, D_j]] = 0\,, \qquad (4)$$

holds for covariant derivatives. (It is sufficient to show this property for the components of a vector field.)

Exercise 38 *Curvature Tensor and Torsion*

Derive the explicit form of the components of the curvature tensor $R^\rho_{\sigma\mu\nu}$ and the torsion tensor $T^\sigma_{\mu\nu}$ in terms of connection coefficients $\Gamma^\mu_{\nu\rho}$ from the relation,

$$[D_\mu, D_\nu] V^\rho = R^\rho_{\sigma\mu\nu} V^\sigma + T^\sigma_{\mu\nu} D_\sigma V^\rho\,. \qquad (5)$$