Exercise 39 *Covariant Derivative.*

Use the definition of the covariant derivative $\nabla_{(.)}$ to express the the connection coefficients,

$$\Gamma^{l'}_{i'k'}\partial_{l'} := \nabla_{\partial_{i'}}(\partial_{k'})\,, \quad \partial_{i'} = \frac{\partial x^j}{\partial x^{i'}}\partial_j\,, \tag{1}$$

in terms of the connection coefficients Γ^l_{mn} in the un-primed coordinate system.

Consider the commutator of covariant derivatives

$$[D_i, D_j]f(x)\,, \tag{2}$$

acting on functions $f(x)$ and show that it is proportional to the torsion tensor.

Consider the action of the covariant derivative on a rank $(2,0)$ tensor and derive the action of the derivative on the components $D_k T^{ij}$,

$$T = T^{ij}\,\partial_i \otimes \partial_j\,, \quad \nabla_k T = (D_k T^{ij})\partial_i \otimes \partial_j\,. \tag{3}$$

Exercise 40 *Curvature Tensor.*

Show the following symmetry properties of the Riemann tensor,

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}\,, \tag{4}$$

$$R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} = 0\,. \tag{5}$$

It is best to work in a locally inertial coordinate system. Argue why the symmetry properties hold in a general coordinate system. How many independent components does the Riemann tensor have in D dimensions. (It is OK to consult the literature for the counting argument.)

Exercise 41 *Implications of the Bianchi Identity.*

Given a metric compatible torsion free connection. Show that the covariant derivative of the tensor $dx^i\partial_i := \sum_i dx^i \otimes \partial_i$ vanishes,

$$\nabla_k\,(dx^i \otimes \partial_i) = 0\,. \tag{6}$$

Give the components of the tensor and show the same statement using component notation, $D_k T^i{}_j$.

Show that the covariant derivative of the inverse metric vanishes,

$$D_i g^{jk} = 0\,. \tag{7}$$

To this end use the definition of the inverse metric $g_{ij}g^{jk} = \delta^k_i$.

Start from the Bianchi identity,

$$D_i R^m{}_{ljk} + D_j R^m{}_{lki} + D_k R^m{}_{lij} = 0\,, \tag{8}$$

and derive that the divergence of the Einstein tensor vanishes,

$$D_i G^{ij} = D_i\big(R^{ij} - \frac{1}{2} g^{ij} R\big) = 0\,. \tag{9}$$

Consider the extension of the Einstein tensor by including the dark-energy term,

$$\big(R^{ij} - g^{ij} R + \Lambda g^{ij}\big)\,, \tag{10}$$

starting with an arbitrary function Λ. Under what conditions does the divergence of this tensor vanish?