Exercise 42 *Red shift in Schwarzschild geometry.*
Consider the Schwarzschild geometry,

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2(\theta) d\phi^2) \,. \tag{1}$$

Compute the length of the space-like line with $r = r_0, \theta = \theta_0, t = t_0$ and $\phi \in [0, 2\pi)$ and interprete the result.

Consider two static observers in the Schwarzschild geometry whose world line is given by $r = r_\pm > 2m, \theta = \theta_0, \phi = \phi_0$ with $r_- < r_+$. Compute the eigentime that passes for the observers as a function of their radial position r.

Consider two radial light rays emitted with delay $\Delta\tau_-$ at r_-. With what delay $\Delta\tau_+$ do the light rays arrive at r_+. Discuss the ratio $\Delta\tau_-/\Delta\tau_+$. What red shift is expected for light; $\nu_+/\nu_- =$?.

Exercise 43 *Einstein equations for Schwarzschild metric.*
Consider the rotationally symmetric and static metric,

$$ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2 d\Omega^2 \,, \tag{2}$$

for undetermined r-dependent functions $\Phi(r)$ and $\Lambda(r)$.

Inserting this metric the Einstein tensor takes the form,

$$G_{00} = \frac{1}{r^2} e^{2\Phi} \frac{d}{dr}[r(1 - e^{-2\Lambda})] \,, \tag{3}$$

$$G_{rr} = -\frac{1}{r^2} e^{2\Lambda}(1 - e^{-2\Lambda}) + \frac{2}{r}\Phi' \,, \tag{4}$$

$$G_{\theta\theta} = r^2 e^{-2\Lambda}[\Phi'' + (\Phi')^2 + \Phi'/r - \Phi'\Lambda' - \Lambda'/r] \,, \tag{5}$$

$$G_{\phi\phi} = \sin^2\theta \; G_{\theta\theta} \,, \tag{6}$$

with the usual short hand notation $f'(r) = df(r)/dr$. Solve the equations for the vacuum $G_{\mu\nu} = 0$ with $r > 0$. Fix integration constants in order to reproduce the expected form of the metric for large values of r where the metric can be approximated using the Newton potential $\phi(r) = G_N \, M/r$ for a point like mass with,

$$ds^2 \sim -(1 + 2\phi(r)) dt^2 + (1 - 2\phi(r)) dr^2 + r^2 d\Omega^2 + \mathcal{O}(1/r^2) \,. \tag{7}$$

Exercise 44 *Towards symmetry transformations in curved spaces.*
Consider the Lie derivative $\mathcal{L}_v$ with respect to a vector v. The aim of the exercise will be to use the Leibnitz rule and linearity for constant coefficients to compute the component form of the derivative. Take as starting point the definition for a function $f(x)$, a vector field $w = w(x)^j \partial_j$ and a one-form $\omega = \omega(x)_i dx^i$,

$$\mathcal{L}_v(f(x)) := v^i \partial_i f(x) \,, \tag{8}$$

$$\mathcal{L}_v(w) := \left(v^i(\partial_i w^j(x)) - (\partial_k v^j(x)) w^k(x)\right) \partial_j \,, \tag{9}$$

$$\mathcal{L}_v(\omega) := \left(v^i(\partial_i \omega_j(x)) + (\partial_j v^k(x)) \omega_k(x)\right) dx^j \,. \tag{10}$$

First of all, show that the Lie derivative of the contraction $\langle \omega, w \rangle = \omega(w) = w^i \omega_i$ is consistent for the above definitions. Consider then a generic tensor,

$$T = T^{i_1,\dots i_k}_{j_1,\dots j_l}\ \partial_{i_1} \otimes \dots \partial_{i_k} \otimes dx^{j_1} \otimes \dots dx^{j_l}\ , \tag{11}$$

and give the component form of its Lie derivative $(\mathcal{L}_v T)^{i_1,\dots i_k}_{j_1,\dots j_l}$.

Is the derivative linear under multiplication of the vector v by a function?