Exercise 45 *Schwarzschild geometry.*
Consider the Schwarzschild geometry,

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2(\theta) d\phi^2) \,. \tag{1}$$

Assume the metric presents a black hole in place of the sun. The aim of this exercise will be to discuss the radial geodesics. Consider the action for a point-like particle and derive the Euler-Lagrange equation for $t(\tau)$. Show that the energy E,

$$E = \left(1 - \frac{2m}{r}\right) \frac{dt}{d\tau} \tag{2}$$

is preserved, where τ denotes the eigentime. Assume a radial geodesic is parametrised with the eigentime parameter and give the differential equation for $dr/d\tau$. How long does it take for a particle falling from earth towards the black hole to cross the horizon at $r_S = 2m$. Assume light-pulses emitted from an object close to the horizon $r = r_S + \epsilon$. Given that the pulses are emitted at fixed interval $\Delta\tau$, how long is the time difference of the arrival of the pulses at a fixed radius away from the horizon $r >> r_S + \epsilon$.

Exercise 46 *Cosmology.*
Verify that the non-vanishing Christoffel symbols of the Robertson-Walker metric $\Gamma^\mu_{\nu\rho}$ are given by the following expressions,

$$\Gamma^0_{ij} = \frac{\dot{R}}{R} g_{ij}, \quad \Gamma^j_{0k} = \frac{\dot{R}}{R}\delta^j_k \,, \quad \Gamma^r_{rr} = \frac{kr}{1 - kr^2} \,, \tag{3}$$

$$\Gamma^r_{\theta\theta} = -r(1 - kr^2) \,, \quad \Gamma^r_{\phi\phi} = -r(1 - kr^2)\sin^2\theta \,, \tag{4}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = \frac{1}{r} \,, \quad \Gamma^\theta_{\phi\phi} = \sin\theta\cos\theta \,, \quad \Gamma^\phi_{\theta\phi} = \cot\theta \,. \tag{5}$$

Exercise 47 *Ergo region.*
Consider an object moving with the speed of light in the angular direction on a path $x^\mu(t) = \{t, r = 0, \theta = \phi/2, \phi(t)\}$. Give the equation of motion $d\phi/dt$ for the object moving in the geometry,

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + ... \tag{6}$$

(The omitted terms are not important for the present considerations.) Show that for $g_{tt} = 0$ the object may remain at fixed position ϕ, although it moves with the speed of light. The location of space time where this is possible is called the ergo sphere. Identify the ergo sphere in the Kerr geometry. Show that somewhat inside the ergo sphere (light-like) objects move only in one direction around the Kerr black hole.