Exercise 28 *Accelerated coordinate system – Rindler space*
Consider two-dimensional flat space time from the point of view of an accelerated observer. Starting point is the flat space metric,

$$ds^2 = -dt^2 + dx^2 \,.$$

Consider next the accelerated coordinate system given by,

$$t = \rho \sinh(a\tau)\,, \quad x = \rho \cosh(a\tau)\,,$$

with $\tau \in R$ and $\rho \in R^+$, the positive real numbers. These coordinates are called Rindler coordinates.

a) Consider the space-time diagram of two dimensional Minkowski space in the $\{t, x\}$ coordinates. Next, discuss the coordinate lines of fixed time τ and fixed position coordinate ρ. Compute the metric in the coordinates τ and ρ. Which part of Minkowski space is covered by the Rindler coordinates? Identify the boundary of the τ-ρ-coordinate system in t-x space time.

b) Consider the world lines of fixed position $\rho = \rho_0$ between $\tau = \pm\tau_0$. Compute the proper time that passes for an observer that moves along these world lines.

c) Compute the geodesic that connects the two events $(\pm\tau_0, \rho_0)$ in the τ-ρ-coordinate system. Consider the geodesic first in the $\{t, x\}$ coordinates and then check that it solves the Euler-Lagrange equations for a free particle in Rindler coordinates. Draw the geodesic and give the proper time that passes for the observer who moves along this segment of the geodesic.

Exercise 29 *Levi-Civita symbol/tensors*
The aim of this exercise is to derive the components of the Levi-Civita tensor.

Starting point is the Levi-Civita symbol,

$$\tilde{\epsilon}_{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{if } (\mu\nu\rho\sigma) \text{ is an even permutation of } (0123), \\ -1 & \text{if } (\mu\nu\rho\sigma) \text{ is an odd permutation of } (0123), \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

a) For a general coordinate transformation $dx^{\mu'} = M^{\mu'}{}_{\mu} dx^{\mu}$ show that the following relation holds,

$$M^{\mu}{}_{\mu'} M^{\nu}{}_{\nu'} M^{\rho}{}_{\rho'} M^{\sigma}{}_{\sigma'}\, \tilde{\epsilon}_{\mu\nu\rho\sigma} = \det(M)\, \tilde{\epsilon}_{\mu'\nu'\rho'\sigma'} \,.$$

b) Give the transformation property of the expression $(\det g)$ under the above coordinate transformation. (Here g stands for the matrix associated to the metric, $\{g_{\mu\nu}\}$.)

c) The components $\epsilon_{\mu\nu\rho\sigma}$ of the Levi-Civita tensor can be written in all coordinate systems in terms of the Levi-Civita symbol and the determinant of the metric,

$$\epsilon_{\mu\nu\rho\sigma} = \big(\det g\big)^a \; \tilde{\epsilon}_{\mu\nu\rho\sigma} \, .$$

What value does the parameter a need to have so that $\epsilon_{\mu\nu\rho\sigma}$ can in fact be interpreted as components of a tensor?

Exercise 30 *Spheres.*

The equation for an 3-dimensional sphere of unit radius embedded in 4 dimensional Euclidean space is given by,

$$ds^2 = \sum_{i=1,4} dx^i dx^i \, ,$$
$$\sum_{i=1,4} x^i x^i = 1 \, ,$$

with a solution in angular variables given by,

$$\begin{aligned} x^1 &= \sin\chi \sin\theta \sin\phi \, , \\ x^2 &= \sin\chi \sin\theta \cos\phi \, , \\ x^3 &= \sin\chi \cos\theta \, , \\ x^4 &= \cos\chi \, , \end{aligned}$$

with $\chi \in [0, \pi]$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$.

a) In the parametric equation of the three-sphere, express the differentials dx^i in terms of the the differentials $d\chi, d\theta$ and $d\phi$. Show that the metric on the three-sphere is given by $ds^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)$.

b) Give the defining equation for a two-dimensional sphere. Give the parametric solution and show that the metric on the sphere is given by $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$.

c) Interprete the subspaces: in the case of the two-sphere; what is the interpretation of the subspace $\theta =$fixed and $\phi \in (0, 2\pi]$. In the case of the three-sphere; what is the interpretation of the subspace defined by $\chi =$fixed, $\theta \in [0, \pi]$ and $\phi \in (0, 2\pi]$.

d) Give the metric on the above subspaces and compute the respective volume forms. Next compute the volumes of the subspaces as a function of θ and χ, respectively. Can you guess the metric and coordinate ranges for a four sphere.