Exercise 31 *One forms*

Consider the following objects df on a differential manifold M, with f a C^∞ function $M \to \mathbf{R}$. These objects are defined as forms which map vectors of the tangent space (V_p) to functions,

$$df : V_p \to C_p^\infty \,, \tag{1}$$

$$df(v) := v^i \partial_i f \,, \tag{2}$$

and $v = v^i \partial_i$, in a local coordinate basis.

Review the vector-space axioms in general. Show that the forms give a vector space for the natural multiplcation and addition rules,

$$(a\,\omega)(v) := a\,\omega(v)\,, \quad (\rho + \omega)(v) := \rho(v) + \omega(v)\,, \quad \text{with} \quad a \in R\,. \tag{3}$$

To this end verify the vector-space axioms for a fixed test-vector $v = v^i \partial_i$.

Show that for the choice of the D functions $x^i(P)_{i=1,D}$ we obtain a set of forms dx^i which are dual to the vectors ∂_i associated to the coordinate basis. (It is recommended to consult a textbook for the derivation if needed.)

Exercise 32 *Tangent vectors*

Consider the following curves in a the coordinate system $\{x^i\}$ of three-dimensional differential manifold that pass through the point p with coordinates $x_p^i = (1, 0, -1)$:

$$C_1 : x^i(\lambda) = (\lambda, (\lambda - 1)^2, -\lambda)$$
$$C_2 : x^i(\mu) = (\cos\mu, \sin\mu, \mu - 1)$$
$$C_3 : x^i(\sigma) = (\sigma^2, \sigma^2 + \sigma^3, \sigma)$$

a) Calculate the components of the three tangent vectors $\frac{d}{dt}$ of these curves (C_1, C_2 and C_3) at the point p in the coordinate basis $\{\partial_i\}$. Give the coordinate independent tangent vectors explicitly.

b) Consider the function $f(x^i) = (x^1)^2 + (x^2)^2 - x^2 x^3$. Calculate the directional derivatives of the function $f(x^i)$ along the three above curves and give $df/d\lambda$, $df/d\mu$ and $df/d\sigma$.

c) Consider the same function $f(x^i)$ as above. This time calculate the differential df: that is evaluate the components ω_i of the one-form $\omega = \omega_i dx^i = df = \partial f/\partial x^i \, dx^i$.

d) Consider the differentials dx^i as the dual vectors to the basis vectors ∂_i with $dx^i(\partial_j) = \delta^i_j$. Evaluate the one-form df on the tangent vectors to the curves C_1, C_2 and C_3. Show that this matches the results of subtask (b).

Exercise 33 *Conserved currents in gauge theories*

Consider the complex valued scalar field $\phi(x) = \phi_r(x) + i\phi_i(x)$ in four dimensions with the action,

$$\mathcal{L}(\phi, A_\mu) = |(\partial_\mu + ieA_\mu)\phi|^2 - m^2|\phi^2| + \mathcal{L}(A_\mu)\,. \tag{4}$$

where $\mathcal{L}(A_\mu) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.

Show that the action is invariant under the simultaneous gauge transformation of both fields, $\phi(x) \to e^{ie\alpha(x)}\phi(x)$ and $A_\mu(x) \to A_\mu(x) - \partial_\mu\alpha(x)$.

Show that the Lagrangian of the gauge potential $\mathcal{L}(A_\mu)$ is individually invariant under the above transformations. Derive the conserved current, by variing the matter action with a localised gauge parameter $\alpha(x)$ and using the equations of motion for the field ϕ.