Exercise 1 *Killing equation.*

The action of the *Lie derivative* along a vector field $V = V^\mu \partial_\mu$ on a tensor field $T = T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} \partial_{\mu_1} \otimes \dots \partial_{\mu_n} \otimes dx^{\nu_1} \otimes \dots dx^{\nu_m}$ is given by,

$$(\mathcal{L}_V T)^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} = V^\sigma \partial_\sigma T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} - \sum_{i=1}^{n} (\partial_\sigma V^{\mu_i}) T^{\mu_1 \dots \mu_{i-1} \sigma \mu_{i+1} \dots \mu_n}_{\nu_1 \dots \nu_m} + \tag{1}$$
$$+ \sum_{i=1}^{m} (\partial_{\nu_i} V^\sigma) T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_{i-1} \sigma \nu_{i+1} \dots \nu_m} .$$

a) Show that all partial derivatives in (1) can be replaced by covariant derivatives without changing the result of the Lie derivative.

b) Show that the Lie derivative acting on the product of two tensors T and S satisfies the Leibniz rule:

$$\mathcal{L}_V (T \otimes S) = (\mathcal{L}_V T) \otimes S + T \otimes (\mathcal{L}_V S)$$

such that,

$$(\mathcal{L}_V (T \otimes S))^{\mu_1 \dots \mu_n \ \alpha_1 \dots \alpha_k}_{\nu_1 \dots \nu_m \ \beta_1 \dots \beta_l} = (\mathcal{L}_V T)^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} S^{\alpha_1 \dots \alpha_k}_{\beta_1 \dots \beta_l} + T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} (\mathcal{L}_V S)^{\alpha_1 \dots \alpha_k}_{\beta_1 \dots \beta_l} .$$

c) Show that the Lie derivative acting on the metric is given by,

$$(\mathcal{L}_V g)_{\mu\nu} = D_\mu V_\nu + D_\nu V_\mu .$$

d) Killing vector fields $K^\mu \partial_\mu$ satisfy the differential equations,

$$D_\mu K_\nu + D_\nu K_\mu = 0,$$

i.e. the Lie derivative of the metric g along a Killing vector vanishes, $\mathcal{L}_V g = 0$. From a Killing vector and the momentum of a point particle, $p^\mu(\tau) = m \frac{dx^\mu}{d\tau}(\tau)$, one can construct the function $Q(\tau) = K_\mu(x^\rho(\tau))\, p^\mu(\tau)$. Show that this function is in fact constant,

$$\frac{dQ}{d\tau} = 0 .$$

if the particle moves along a geodesic.

Exercise 2 *Geodesic deviation.*

Consider a one-parameter family of geodesics $\gamma_s^\mu(t)$, giving for each value of s a geodesic parametrized with affine parameter t. The tangent vector T^μ to the geodesics and a vector field measuring the deviation between nearby geodesics S^μ are given by,

$$T^\mu = \frac{\partial \gamma_s^\mu(t)}{\partial t}\,, \quad S^\mu = \frac{\partial \gamma_s^\mu(t)}{\partial s}\,, \tag{2}$$

respectively. Furthermore the relative velocity V^μ of geodesics is given by,

$$V^\mu = (\nabla_T S)^\mu = T^\rho \nabla_\rho S^\mu \tag{3}$$

and the relative acceleration A^μ by,

$$A^\mu = (\nabla_T V)^\mu = T^\rho \nabla_\rho V^\mu\,. \tag{4}$$

Show that the relative acceleration between geodesics is proportional to the curvature of space,

$$A^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma\,. \tag{5}$$

Please consult the standard literature to construct the derivation of this result (e.g. S. Carroll's book).

Exercise 3 *Gauge transformations*

Consider a small perturbation of flat spacetime,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,, \tag{6}$$

and the infinitesimal diffeomorphisms,

$$f: \quad f^\mu(x) = x^\mu + \xi^\mu(x)\,, \tag{7}$$

ignoring quadratic terms in the transformation $\xi^\mu(x)$, fluctuations $h_{\mu\nu}(x)$ as well as mixed terms.

Compute the pull back of the metric using the above diffeomorphism and show that the pull-back metric can be interpreted as a transformation of the metric fluctuation,

$$f^*(g)_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu =: \eta_{\mu\nu} + h_{\mu\nu} + \delta h_{\mu\nu}\,. \tag{8}$$

It turns out that the above transformation $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ defines a gauge transformation of the metric fluctuation since, on the one hand, it transforms solutions to the equations of motion (Einstein equations) to new solutions and, on the other hand, the transformation parameters $\xi^\mu(x)$ can be varied locally being (rather) generic vector fields. (Note the typical definition $\xi_\mu(x) := \eta_{\mu\nu}\xi^\nu$.)

As a first step towards showing the gauge invariance of the linearized Einstein equations verify that the linearized curvature tensor,

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\rho \partial_\nu h_{\mu\sigma} + \partial_\sigma \partial_\mu h_{\nu\rho} - \partial_\sigma \partial_\nu h_{\mu\rho} - \partial_\rho \partial_\mu h_{\nu\sigma}) \tag{9}$$

is invariant under the above gauge transformations of the metric fluctuation.