Exercise 27 *Penrose process of charged particle in RN geometry*

The equations of motion for a charged massive particle in a gravitational and electromagnetic background is given by,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = \frac{e}{m}F^\mu{}_\nu\frac{dx^\nu}{d\tau}\,. \tag{1}$$

Consider the Reissner Nordstrom (RN) geometry of the electrically charged black hole,

$$ds^2 = -\Delta dt^2 + \Delta^{-1}dr^2 + r^2 d\Omega^2\,, \tag{2}$$

$$\Delta = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\,. \tag{3}$$

a) Show that the Lie-derivative of the electromagnetic potential $A = Q/r dt$ vanishes with respect to the time-like Killing vector field ∂_t.

b) Show that the scalar expression

$$E(\tau) = -m\left(\frac{dx^\mu}{d\tau} + \frac{e}{m}A^\mu\right)K_\mu\,, \tag{4}$$

is conserved; $\dot{E}(\tau) = 0$ with $K^\mu = (\partial_t)^\mu$ the components of the time like Killing vector.

c) Compute the energy of a particle at rest at infinity. Compute the energy of a charged particle on the horizon. Finally, consider a charged particle $eQ < 0$ falling towards the horizon, but emitting a neutral massive particle towards infinity before the charge crosses the horizon. What amount of energy does the neutral particle carry at infinity? Give the change of energy δM and charge δQ of the black hole in this process.

d) Compute the change of the black hole area as a function of change of mass and charge by absorbing the infalling charge. Check that the area of the black hole horizon is not decreasing?

Exercise 28 *Rinder computations*

Construct the following quantities in Rindler space,

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2)\,, \tag{5}$$

with a a constant.

a) Compute the Laplace operator acting on a scalar field $\Delta\phi = D^\mu D_\mu \phi$.

b) Derive the explicit form of the inner product of two complex scalar fields $\phi_{1,2}$ given by,

$$(\phi_1, \phi_2) := -i \int_\Sigma (\phi_1 \nabla_\mu \phi_2^* - \phi_2^* \nabla_\mu \phi_1) n^\mu \sqrt{\gamma} d\xi \,. \tag{6}$$

with Σ the spacelike surface defined by $\eta = 0$ and n^μ the unit normal vector to the surface and γ the determinant of the pull back of the metric. Show that the inner product is time independent, i.e. $\partial_\eta(\phi_1, \phi_2) = 0$, for solutions $\phi_{1,2}$ to the Laplace equation.

c) Consider the below coordinate transfromations and derive the form of the Killing vector ∂_η in Minkowski coordinates.

The coordinate transfromation from Rindler to Minkowski coordinates is given by,

$$t = \frac{1}{a} e^{a\xi} \sinh(a\eta) \,, \quad x = \frac{1}{a} e^{a\xi} \cosh(a\eta) \,. \tag{7}$$

Identify the Killing horizon of the vector field ∂_η.