Exercise 4 *Scalar field theory*

As an attempt at a theory of gravity in terms of a scalar field $\phi(x)$ on Minkowski spacetime, consider the action of a point particle coupled to ϕ:

$$\begin{aligned}
S &= S_\phi + S_{\rm mat} + S_{\rm int}, \\
S_\phi &= -\frac{1}{2\kappa}\int d^4x\, \partial_\mu\phi(x)\partial^\mu\phi(x), \\
S_{\rm mat} &= m\int d\tau = m\int d\lambda\sqrt{-\frac{dx^\mu(\lambda)}{d\lambda}\frac{dx_\mu(\lambda)}{d\lambda}}, \\
S_{\rm int} &= -\int d^4x T^\mu_{{\rm mat},\mu}(x)\phi(x), \\
T^{\mu\nu}_{\rm mat}(z) &= m\int \frac{dx^\nu(\tau)}{d\tau}\frac{dx^\mu(\tau)}{d\tau}\delta^{(4)}(z^\rho - x^\rho(\tau))\, d\tau\,.
\end{aligned}$$

Here λ is an arbitrary parametrization of the trajectory and τ is the proper time, $d\tau^2 = -\frac{dx^\mu(\lambda)}{d\lambda}\frac{dx_\mu(\lambda)}{d\lambda}d\lambda^2$.

a) Derive the field equation of the field ϕ. Discuss the limit of a static matter distribution, $T^\mu_{{\rm mat},\mu}(x) \approx T^0_{{\rm mat},0}(\vec{x})$.

b) Show that the interaction of the field and the point particle can be written in the form

$$S_{\rm int} = m\int d\tau\phi(x(\tau)).$$

c) Derive the equation of motion for the trajectory $x^\mu(\lambda)$. Show that for the choice $\lambda = \tau$ the equation can be written in the form

$$\frac{d^2}{d\tau^2}x^\mu = -\left(\delta^\mu_\nu + \frac{dx^\mu(\tau)}{d\tau}\frac{dx^\nu(\tau)}{d\tau}\right)\partial_\nu\Phi(x) \tag{1}$$

with $\Phi(x) = \ln(1+\phi(x))$.

d) Argue that the right-hand side of eq. (1) defines a consistent Minkowski force. Why is an equation of motion of the form $\frac{d^2}{d\tau^2}x^\mu = -\partial^\mu\phi(x)$ inconsistent?

Exercise 5 *Gauge fixing.*

Consider the following "gauge fixed" Lagrangian for linearized gravity:

$$\mathcal{L} = \mathcal{L}_{\mathrm{FP}} + \mathcal{L}_{\mathrm{gf}}, \tag{2}$$

$$\mathcal{L}_{\mathrm{FP}} = -\frac{1}{2}\left[\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} - \partial_\mu h \partial^\mu h + 2\partial_\mu h^{\mu\nu}\partial_\nu h - 2\partial_\mu h^{\mu\rho}\partial_\nu h^\nu_\rho\right] + \frac{\kappa}{2}h_{\mu\nu}T^{\mu\nu},$$

$$\mathcal{L}_{\mathrm{gf}} = -(\partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h)^2.$$

a) Show that the action obtained from (2) can be written in the form

$$S = -\frac{1}{2}\int d^4x M_{\mu\nu\rho\sigma}\partial^\alpha h^{\mu\nu}\partial_\alpha h^{\rho\sigma}$$

with

$$M_{\mu\nu\rho\sigma} = \frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma})$$

b) Show that the field equations derived from (2) reproduce the field equations in harmonic gauge,

$$\Box\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right) = -\frac{\kappa}{2}T_{\mu\nu}$$

c) Show that the function

$$G^{\mu\nu\rho\sigma}(x - x') = \int \frac{d^4p}{(2\pi)^4}\frac{M^{\mu\nu\rho\sigma}}{p^2}e^{-\mathrm{i}p_\mu(x-x')^\mu}$$

is a Green function for the field equations in harmonic gauge, i.e.

$$M_{\mu\nu\rho\sigma}\Box_x G^{\rho\sigma\alpha\beta}(x - x') = -\frac{1}{2}(\delta^\alpha_\mu\delta^\beta_\nu + \delta^\beta_\mu\delta^\alpha_\nu)\delta^4(x - x').$$

Note that the tensor on the right-hand side is the identity matrix on the space of tensors symmetric under the exchange of $\mu \leftrightarrow \nu$ and $\alpha \leftrightarrow \beta$.