Übungen ART II SoSe 2015

Exercise 6 *Transverse-traceless gauge*

The polarization tensors of a plane gravitational wave with wave-vector k^μ in *transverse-traceless gauge* satisfy the conditions

$$\epsilon'^{\mu}{}_{\mu}(k) = 0\,, \qquad \epsilon'^{\mu\nu}(k)u_\nu = 0, \tag{1}$$

with a time-like unit vector u_ν, $u^2 = -1$, which satisfies $(u \cdot k) \neq 0$. Show that any polarization tensor satisfying the Lorenz-gauge condition $\epsilon^{\mu\nu}(k)k_\nu = 0$ can be brought to transverse-traceless gauge using a gauge transformation of the form

$$\epsilon'^{\mu\nu}(k) = \epsilon^{\mu\nu}(k) - \mathrm{i}(k^\mu \xi^\nu(k) + \xi^\mu(k)k^\nu) + \mathrm{i}\eta^{\mu\nu}(k \cdot \xi(k)).$$

a) Write down the conditions on the gauge functions ξ^μ arising from the equations (1) and argue that they can be satisfied using the Ansatz

$$\xi^\mu(k) = \xi_0 u^\mu + \xi_+ k^\mu + \xi_\perp \epsilon^{\mu\nu}(k)u_\nu.$$

b) Show that ξ_0 is determined from the condition for the tracelessness of $\epsilon'^{\mu\nu}$.

c) Show that $\xi_\perp$ and ξ_+ can be chosen so that the second condition in (1) is satisfied.

Exercise 7 *Particle motion in gravitational waves*

Consider a gravitational wave, which is described in a coordinate system where it satisfies the transverse-traceless gauge. In this coordinate system, the invariant line element takes the form

$$ds^2 = -dt^2 + dz^2 + (1 + h_+(t-z))dx^2 + (1 - h_+(t-z))dy^2, \tag{2}$$

with some function $h_+(t-z)$.

a) Consider a particle at rest at the point $\vec{x}$ at time $t = 0$. Evaluate the geodesic equation at time $t = 0$ to conclude that the particle stays at rest in the presence of the gravitational wave in the transverse-traceless coordinate system.

b) Show that the proper distance $\Delta s = \int_{x_0}^{x_1} \sqrt{ds^2}$ between the two space-time points $x_0^\mu = (t, x, 0, 0)$ and $x_1^\mu = (t, x + L, 0, 0)$ for small L is approximately given by

$$\Delta s \approx \left(1 + \frac{1}{2}h_+(t)\right) L,$$

allowing to measure the effect of the gravitational wave.

Exercise 8 *Toy model of a gravitational wave detector*

As a simple model for a gravitational wave detector, consider two point masses connected by a spring. The equation of motion for the separation S^i of the two masses is given by that of a damped harmonic oscillator in an external force given in terms of the field h_{ij}^{TT} of the gravitational wave:

$$\ddot{S}^i + 2\gamma\dot{S}^i + \omega_0^2 S^i = \frac{1}{2}\ddot{h}_{ij}^{TT} S^j$$

with the damping rate γ. This expression can either be derived in the transverse-traceless frame taking the result of Ex. 7 b) for the proper length of the spring into account, or in a freely-falling frame using the result for the effective Newtonian force derived in the lecture. For the example of a plane gravitational wave propagating along the z-axis with angular frequency ω and for two masses moving along the x-axis, the equation can be approximated as

$$\ddot{S} + 2\gamma\dot{S} + \omega_0^2 S = -\frac{\omega^2}{2} h_+ S_0 \cos\omega t, \tag{3}$$

where S_0 is the unstretched length of the spring and the constant h_+ is the amplitude of the gravitational wave.

a) The solution to eq. (3) is of the form $S = R\cos(\omega t + \varphi)$. Use standard results for the forced, damped harmonic oscillator to compute the amplitude R for the resonant case $\omega = \omega_0$.

b) If the two masses are initially at rest, the energy of the oscillator is given by

$$E = \frac{m}{4}(\dot{S}^2 + \omega_0^2 S^2)$$

Compute the average energy $\langle E \rangle$ over one period $2\pi/\omega$.

c) How large are R and $\langle E \rangle$ for the values $\omega = \omega_0 = 1$ kHz, $S_0 = 2$ m, $m = 2 \times 10^3$ kg, $h_+ = 10^{-20}$ and the "quality factor" of the oscillator $Q = \omega/2\gamma = 10^6$? For realistic examples of resonance detectors for gravitational waves you can consult websites of the projects AURIGA (`http://www.auriga.lnl.infn.it/`) and miniGRAIL (`http://www.minigrail.nl/`).