Exercise 9 *Transverse-traceless projection of gravitational waves*

A plane gravitational wave in harmonic gauge, $\bar{h}^{\mu\nu}(x)$, with wave-vector $k^\mu = k(1, n^i)$ can be projected on the transverse-traceless components according to

$$h_{ij}^{TT} = \Lambda_{ij,kl}(\vec{n})\bar{h}_{kl} = (P\bar{h}P)_{ij} - \frac{1}{2}P_{ij}\mathrm{tr}P\bar{h}$$

with the projector

$$\Lambda_{ij,kl}(\vec{n}) = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\,, \qquad \text{with} \qquad P_{ij} = \delta_{ij} - n_i n_j.$$

In this exercise, upper and lower space-like indices i are not distinguished.

a) Verify that the $\Lambda_{ij,kl}$ indeed form a projector, i.e.

$$\Lambda_{ij,kl}\Lambda_{kl,mn} = \Lambda_{ij,mn}$$

b) Verify that the projected gravitational wave is indeed in the transverse-traceless gauge, i.e. it satisfies

$$h_{ii}^{TT} = 0, \qquad\qquad k_i h_{ij}^{TT} = 0$$

c) The gravitational wave amplitude in the quadrupole approximation is given by

$$h_{ij}^{TT}(t, \vec{x}) = \frac{2G}{r}\Lambda_{ij,kl}(\vec{n})\ddot{Q}_{kl}(t-r)$$

with $\vec{x} = r\vec{n}$. For a gravitational wave propagating along the z-direction, $\vec{n} = (0, 0, 1)$,

$$h_{ij}^{TT}(t, r) = (h_\times(t, r)\epsilon_{ij,\times} + h_+(t, r)\epsilon_{ij,+})$$

with the polarization tensors

$$\epsilon_{ij,+} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad\qquad \epsilon_{ij,\times} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

show that the amplitudes of the two polarizations are given by

$$h_+(t, r) = \frac{G}{r}\left(Q_{11}(t-r) - Q_{22}(t-r)\right), \qquad h_\times(t, r) = \frac{2G}{r}Q_{12}(t-r). \qquad (1)$$

Exercise 10 *Production of gravitational waves*

a) Consider two non-relativistic point masses m_1 and m_2 at the locations $\vec{x}_1(t)$ and $\vec{x}_2(t)$. Show that the quadrupole tensor

$$Q_{ij}(t) = \int d^3x\, \rho(t,\vec{x}) \left(x_i x_j - \frac{1}{3}\delta_{ij}\vec{x}^2 \right) ,$$

where ρ is the mass density, can be written in the form

$$Q_{ij} = M(X_i X_j - \frac{1}{3}\delta_{ij}\vec{X}^2) + \mu(x_i x_j - \frac{1}{3}\delta_{ij}\vec{x}^2)$$

with the centre of mass and relative coordinates, $\vec{X} = (m_1\vec{x}_1 + m_2\vec{x}_2)/(m_1+m_2)$ and $\vec{x} = \vec{x}_1 - \vec{x}_2$, respectively. The total and reduced mass are defined by $M = m_1 + m_2$ and $\mu = m_1 m_2/(m_1 + m_2)$.

b) Compute the quadrupole tensor for two point masses oscillating along the x-axis with the relative coordinates $\vec{x}(t) = (L + a\cos\omega t, 0, 0)$ and the centre-of-mass coordinate $\vec{X} = 0$. Use Eq. (1) in exercise 9 to compute the components of the gravitational wave along the z-axis.

c) Repeat part b) for two point masses on a circular orbit in the $x - y$ plane with the relative coordinates $\vec{x}(t) = (R\sin\omega t, R\cos\omega t, 0)$ and the centre-of-mass coordinate $\vec{X} = 0$.

Exercise 11 *Energy-momentum tensor*

Consider the action for the free electromagnetic field ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu$).

$$S(A_\mu, g_{\mu\nu}) = -\frac{1}{4}\int_{\mathbf{R}^4} \sqrt{-g} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} d^4x$$

a) Compute the *canonical* energy-momentum tensor of the electromagnetic field

$$T^{\mu\nu}_{\text{can}} = -\frac{\partial\mathcal{L}}{\partial\partial_\mu A_\rho}\partial^\nu A_\rho + g^{\mu\nu}\mathcal{L}$$

in Minkowski space, $g^{\mu\nu} = \eta^{\mu\nu}$.

b) Compute the *symmetric* energy-momentum tensor according to the definition

$$\delta_g S(A_\mu, g_{\mu\nu}) = \frac{1}{2}\int T^{\mu\nu}_{\text{sym}}\delta g_{\mu\nu}\sqrt{-g}dx^4$$

and let $g^{\mu\nu} \to \eta^{\mu\nu}$ subsequently. Recall the relations

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu} , \qquad \delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma} .$$

c) Show that the difference of the results of part a) and b) can be written in the form

$$T^{\mu\nu}_{\text{sym}} - T^{\mu\nu}_{\text{can}} = \partial_\rho\Sigma^{\rho\mu\nu} , \qquad \text{with} \qquad \Sigma^{\rho\mu\nu} = -\Sigma^{\mu\rho\nu},$$

if the field equation $\partial_\mu F^{\mu\nu} = 0$ is satisfied.