Übungen ART II SoSe 2015

Exercise 12 *Energy-momentum tensor*

The energy-momentum tensor of gravitational waves in the harmonic gauge, $\partial_\alpha \bar{h}^{\alpha\beta}(x) = 0$, is given by

$$\langle T_h^{\mu\nu} \rangle = \frac{1}{32\pi G} \langle \partial^\mu \bar{h}^{\alpha\beta} \partial^\nu \bar{h}_{\alpha\beta} - \frac{1}{2} \partial^\mu \bar{h} \partial^\nu \bar{h} \rangle,$$

where the spatial average $\langle F \rangle$ has the property that $\langle \partial_\mu F(x) \rangle \approx 0$, so that integration by part can be performed without boundary terms. Show that the energy-momentum tensor is invariant under the gauge transformation

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu}(\partial \cdot \xi).$$

Exercise 13 *Graviton helicity*

Consider a plane gravitational wave in z-direction,

$$h_{ij}(t, z) = (h_\times \epsilon_{ij,\times} + h_+ \epsilon_{ij,+}) \cos(k(t - z)),$$

with the same polarization tensors as given in exercise 9. The effect of a rotation around the z-axis by an angle θ, $R_{ij}(\theta)$, can be written in the form

$$R_{ik}(\theta) R_{jl}(\theta) h_{kl}(t, z) = \left(h'_\times \epsilon_{ij,\times} + h'_+ \epsilon_{ij,+}\right) \cos(k(t - z)),$$

where upper and lower indices are not distinguished and a summation over repeated indices is implied. Compute the coefficients h'_+ and $h'_\times$ and show that the combinations $h_\pm = h_\times \pm \mathrm{i} h_+$ transform as

$$h'_\pm = e^{2\mathrm{i}\theta} h_\pm.$$

This is the behaviour of *helicity-two* states under a rotation around the direction of motion.

Exercise 14 *Photon propagator*

The momentum-space propagator of the photon, $D_{F,\nu\rho}(k)$ is naively defined as the inverse of the wave operator

$$\mathcal{D}^{\mu\nu}(k) = g^{\mu\nu}k^2 - k^\mu k^\nu \,,$$

i.e.

$$\mathcal{D}^{\mu\nu}(k) D_{F,\nu\rho}(k) = \mathrm{i} g^\mu_\rho \,. \tag{1}$$

This exercise discusses various aspects of the fact that the wave operator is not invertible, so that the propagator has to be defined using gauge fixing.

a) Make an Ansatz for the propagator $D_{F,\nu\rho}(k)$ and try to solve the equation (1) directly.

b) Discuss the general solutions to the free Maxwell equations in momentum space,

$$\mathcal{D}^{\mu\nu}(k)\tilde{A}_\nu(k) = 0.$$

Consider first the space-like momentum $k = (0, 0, 0, 1)$. Write the Fourier transform as a linear combination of polarization components and a pure gauge contribution,

$$\tilde{A}(k) = \tilde{a}_t(k)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \tilde{a}_x(k)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \tilde{a}_y(k)\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \tilde{\alpha}(k)k\,,$$

for arbitrary functions $\tilde{a}_t(k), \tilde{a}_x(k), \tilde{a}_y(k), \tilde{\alpha}(k)$. Show that there is no solution to the field equations other than the pure-gauge contribution, $\tilde{\alpha}(k)k_\mu$. Show the analogous statement for time-like momenta $k = (1, 0, 0, 0)$. What is the rank of the operator $\mathcal{D}^{\mu\nu}(k)$ for these space-like and time-like momenta, respectively?

c) Consider next a light-like momentum vector $k = (1, 0, 0, 1)$ and the decomposition

$$\tilde{A}(k) = \tilde{a}_x(k)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \tilde{a}_y(k)\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \tilde{\beta}(k)\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} + \tilde{\alpha}(k)k\,.$$

Discuss the solutions to the field equations. What is the rank of the operator $\mathcal{D}^{\mu\nu}(k)$ in momentum space for these like-like momenta?

d) Consider now the gauge-fixed wave-operator

$$\mathcal{D}^{\mu\nu}_\xi(k) = g^{\mu\nu}k^2 - \left(1 - \frac{1}{\xi}\right)k^\mu k^\nu\,.$$

Discuss the rank of this operator for the various cases and construct the propagator $D_{F,\nu\rho}(k)$.