Exercise 15 *Graviton exchange*

The scattering amplitude for a scattering process $\phi_1(p_1)\phi_2(p_2) \to \phi_1(k_1)\phi_2(k_2)$ of two distinguishable particles $\phi_{1/2}$ is given by

$$\mathrm{i}\mathcal{M} = \left(-\mathrm{i}\frac{\kappa}{2}\right)^2 T^{(1)}_{\mu\nu}(p_1, k_1, q) D_F^{\mu\nu\rho\sigma}(q) T^{(2)}_{\rho\sigma}(p_2, k_2, q), \tag{1}$$

with the momentum transfer $q = k_1 - p_1 = p_2 - k_2$ and where $T^{(i)}_{\mu\nu}(p_i, k_i, q)$ is the Fourier transform of the energy-momentum tensor of particle ϕ_i. The graviton propagator is given by

$$D_F^{\mu\nu\rho\sigma}(q) = \frac{-\mathrm{i}}{2}\frac{\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}}{q^2}.$$

a) For $q^\mu = (\omega, 0, 0, k)$, use the continuity equation for the energy-momentum tensors, $q_\mu T^{(i),\mu}_\nu(p_i, k_i, q) = 0$, to eliminate the components $T^{(i)}_{33}(p_i, k_i, q)$ in the amplitude.

b) Show that the amplitude can be written in the form

$$\mathcal{M} \sim T^{(1)}_{\mu\nu}(p_1, k_1, q)\frac{\epsilon^{\mu\nu}_{+}(q)\epsilon^{\rho\sigma}_{+}(q) + \epsilon^{\mu\nu}_{\times}(q)\epsilon^{\rho\sigma}_{\times}(q)}{k^2 - \omega^2} T^{(2)}_{\rho\sigma}(p_2, k_2, q) + \ldots$$

with the graviton polarization tensors $\epsilon^{\mu\nu}$. The terms indicated by dots do not have a pole at $\omega^2 = k^2$.

Exercise 16 *Gravitational interaction of photons*

In this problem we will discuss scattering of two photons, $\gamma(p_1)\gamma(p_2) \to \gamma(k_1)\gamma(k_2)$ in the limit of small momentum transfer, $q^2 \equiv (k_1 - p_1)^2 \approx 0$. The dominant contribution to the scattering amplitude in this limit is of the form of Eq. (1). The photon momenta satisfy $p_i^2 = k_i^2 = 0$ and the photon polarization vectors are transverse, $p_i \cdot \epsilon(p_i) = 0$, and $k_i \cdot \epsilon^*(k_i) = 0$. The Fourier transform of the energy-momentum tensor of photons is given by

$$\begin{aligned} T^{\mu\nu}(p, k, q) =& (k^\mu \epsilon^{*,\alpha}(k) - k^\alpha \epsilon^{*,\mu}(k))(p^\nu \epsilon_\alpha(p) - p_\alpha \epsilon^\nu(p)) \\ &+ (p^\mu \epsilon_\alpha(p) - p_\alpha \epsilon^\mu(p))(k^\nu \epsilon^{*,\alpha}(k) - k^\alpha \epsilon^{*,\nu}(k)) \\ &- \eta^{\mu\nu}\left[(k \cdot p)(\epsilon^*(k) \cdot \epsilon(p)) - (\epsilon^*(k) \cdot p)(k \cdot \epsilon(p))\right]. \end{aligned} \tag{2}$$

a) Show that the energy-momentum tensor (2) is traceless and conserved,

$$T^\mu{}_\mu(p, k, q) = 0, \qquad q_\mu T^{\mu\nu}(p, k, q) = 0$$

with $q = k - p$.

b) Show that the energy-momentum tensor (2) in the limit $q^\mu \to 0$, i.e. $p^\mu \sim k^\mu$ takes the form

$$T^{\mu\nu}(p, k, q) \approx 2p^\mu p^\nu(\epsilon(p) \cdot \epsilon(k)).$$

c) Compute the scattering amplitude (1) in the $q^\mu \to 0$ limit. Is there a gravitational attraction of two parallel light-rays ($\vec{p}_1 \propto \vec{p}_2$) and two anti-parallel light-rays ($\vec{p}_1 \propto -\vec{p}_2$)?