Exercise 17 *Gravity scales*

The aim of this exercise is to derive a number of characteristic scales in black-hole theory.

a) Assume a spherical body of mass M. Give the formula for its Schwarzschild radius and its Compton wave length. For which value of the mass do Schwarzschild radius and Compton wave length match.

b) Translate the above result into electron volt, a length scale (cm) and a time scale (s). These scales mark a regime where mixed quantum and relativistic effects become important.

c) Compute the Schwarzschild radius of the sun, a neutron star (with 1.4 times the solar mass) and a galaxy cluster consisting of a billion stars. As an estimate assume these black holes to have constant density all the way from their center to their horizon. How does the density compare to that of water?

d) Look up the formula for the Hawking temperature of a black hole in cgs units. Describe the black hole radiation assuming black holes are perfect black bodies; give the area of the black hole and apply the Stefan-Boltzmann law in order to obtain the energy radiation rate dE/dt. Determine the life time of black holes as a function of their mass. Give the mass, temperature and life time of a proton-size black hole. Compare this time scale to the present age of the universe. Give the characteristic scales for a solar weight ($M_{sol} = 10^{33} g$) black hole.

Exercise 18 *Relations for the surface gravity of Killing horizons*

In the following assume a vector field ξ that is hypersurface orthogonal;

$$\xi_{[\alpha} \nabla_\beta \xi_{\gamma]} = 0\,, \tag{1}$$

and in addition is a Killing vector field with $\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$. The surface gravity κ_ξ, associated to a Killing horizon on the stationaly limit surface $S = -\xi^\alpha \xi_\alpha = 0$ can be defined by,

$$\xi^\beta \nabla_\beta \xi_\alpha = \kappa_\xi \, \xi_\alpha\,. \tag{2}$$

That is it is given by the non-affinity of the geodesic equation of the Killling field.

a) Show that the following alternative definition of the surface gravity holds: $(\kappa_\xi)^2 = -1/2(\nabla^\alpha \xi^\beta)(\nabla_\alpha \xi_\beta)$.
Instructions: Write out the condition for hypersurface orthogonality and simplify the sum using the Killing equation. Finally contract the remaining expression with either the left or the right hand-side the Killing equation $\nabla^\alpha \xi^\beta = -\nabla^\beta \xi^\alpha$.

b) Show that the following simpler definition holds as well, $\nabla_\alpha S = 2\kappa_\xi \xi_\alpha$.
Instructions: evaluate the left-hand side using the Killing equation and then plug in the original definition of the surface gravity, that is equation (2).

c) Use the above relation to compute the surface gravity of the event horizon of the Schwarzschild solution. The event horizon is the Killing horizon at $ds^2(\xi, \xi) = 0$ of the time-like Killing vector field $\xi = \partial_t$ for the metric,

$$ds^2 = (1 - 2GM/r)dt^2 - (1 - 2GM/r)^{-1}dr^2 + r^2 d\Omega^2 \,. \tag{3}$$