**Exercise 19** *Conserved charge of black holes*

Consider the Komar integral associated with the Killing vector field $\xi$,

$$Q_\xi = \frac{1}{4\pi G}\int dx^2 \sqrt{\gamma^{(2)}} n_\alpha \sigma_\beta \nabla^\alpha \xi^\beta \,, \tag{1}$$

for the Schwarzschild black hole. Here the integral is performed over a two-sphere at spatial infinity.

The quantities in the integral are given as follows: $n_\alpha$ denotes components of the time like unit one-form, $\sigma_\beta$ denotes components of the radial unit one-form. $\xi$ denotes a Killing vector field and $\gamma^{(2)}_{ij}$ stands for the pull back of the metric to the two sphere at spatial infinity which is given by $t = fixed$ and large radial coordinate $r \to \infty$.

Compute the conserved charges corresponding to the Killing vectors $\xi = \partial_t$ and $\xi = \partial_\phi$ of the geometry.

**Exercise 20** *Surface gravity and Hawking temperature.*

The Hawking temperature of a black hole is given in terms of the surface gravity $\kappa_\xi$ of the event horizon, $T_H = \kappa_\xi/2\pi$. The aim of this exercise is to collect evidence that the surface gravity is constant along the event horizon implying a constant temperature over the *surface* of the black hole

Use the definition of the surface gravity of a Killing horizon of the Killing vector field $\xi$,

$$\xi^\beta \nabla_\beta \xi_\alpha = \kappa_\xi \, \xi_\alpha \,, \tag{2}$$

and compute the directional derivative of $\kappa_\xi$ along the Killing field $\xi$ following the below steps:

a) The Killing horizon coincides with the location where the Killing vector is light like $S = -\xi^\alpha \xi_\alpha = 0$. Demonstrate that the Killing vector field is tangent to the Killing horizon using the Killing equation.

b) Compute the directional derivative of equation (2) in order to show that $\kappa_\xi$ does not change along the tangent direction $\xi$. Remark: use the property of Killing vectors, $\nabla_\gamma \nabla_\beta \xi^\alpha = R^\alpha_{\beta\gamma\delta} \xi^\delta$ in order to evaluate the covariant derivatives.

**Exercise 21** *Coordinate systems in Schwarzschild geometry*

The aim of the exercise is to find the geodesic completion of the metric,

$$ds^2 = -x^2 dt^2 + dx^2 \,, \tag{3}$$

defined for $-\infty < t < \infty$, $0 < x < \infty$ with a coordinate singularity at $x = 0$. Instructions:

a) Transform to light-cone coordinates $(u, v)$ of the ingoing and outgoing light rays.

b) Introduce an affine parametrization for the light rays. To this end use that $\partial_t$ is a Killing vector and that the energy $E$ is conserved along light rays with affine parametrization (here the parameter is called $\lambda$); $k^a = dx^a/d\lambda$ and $E := -g_{ab}k^a(\partial_t)^b$. this can be used to tranform the light-cone coordinates to *affine* light-cone coordinates $(U, V)$.

c) Give the metric in time and space coordinats $T = (U + V)/2, X = (V - U)/2$ and draw the space time diagram. Which part of the extended spacetime corresponds to the original space time.