Exercise 22 *Field strength with spherical symmetry*

The aim of this exercise is to construct the electro-magnetic field-strength tensor for a point like electro-magnetic charge; with charge vector $\vec{q} = (Q_e, Q_m)$. Furthermore, the resulting field strength should be expressed in spherical coordinates for the space time,

$$ds^2 = -f(r,t)dt^2 + f(r,t)^{-1}dr^2 + r^2 d\Omega^2 \,. \tag{1}$$

Solve the following subtasks:

a) Start from the one-form potential for a spherical point-like charge $\phi(r; Q) = Q/r\, dt$. Use the potential to calculate the field strength $F_e = d\phi(r, Q_e)$ as well as the dual field strength $\tilde{F}_m = d\phi(r; Q_m)$ of the magentic potential.

b) Next, superimpose the field strength $\tilde{F}_m$ to the electric field strength F_e to obtain the electro-magnetic field strength $F = F_e + *\tilde{F}_m$ for a electro-magnetic charge vector $\vec{q}$.

c) Use Stoke's theorem to compute the electric and magnetic charges from surface integrals. The Maxwell equations for a mix of electric and magnetic charges is given by,

$$dF = *j_m \,, \quad d\tilde{F} = *j_e \,, \quad \tilde{F} := *F \,. \tag{2}$$

Exercise 23 *Horizons in Kerr geometry*

The event horizons in the Kerr geometry are Killing horizons of the Killing field $\chi = \partial_t + \Omega_H^{\pm}\partial_\phi$ where $\Omega_H^{\pm}$ is constant. Use the definition for a Killing horizon $f(x) = ds^2(\chi, \chi) = 0$ as well as the condition that the surface is light like in order to determine the constants $\Omega_H^{\pm}$ and the equation $r = r_{\pm}(a, M)$ defining the event horizons.

The Kerr metric is given by,

$$\begin{aligned}
ds^2 &= -\left(1 - \frac{2GMr}{\rho^2}\right) dt^2 - \frac{2GMar\sin^2\theta}{\rho^2}\,(dtd\phi + d\phi dt) + \frac{\rho^2}{\Delta}dr^2 + && (3)\\
&\quad +\rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right] d\phi^2 \,, && (4)\\
\Delta(r) &= r^2 - 2GMr + a^2 \,, && (5)\\
\rho^2(r,\theta) &= r^2 + a^2\cos^2\theta \,, && (6)
\end{aligned}$$

with the Killing vectors being ∂_t and ∂_ϕ.

Exercise 24 *Hyper-surface orthogonal Killing vectors*

The condition for a vector field $\xi^\mu\partial_\mu$ to be hyper-surface orthogonal is given by $\xi_{[\mu}\nabla_\nu\xi_{\rho]} = 0$ or similarly in form notation by $\zeta \wedge d\zeta = 0$ with $\zeta = \xi_\mu dx^\mu$.

a) Are the above conditions in fact identical? Under what conditions?

b) Verify that the Killing vector field ∂_t is hyper-surface orthogonal in the Schwarzschild geometry. Is the Killing field ∂_ϕ hyper-surface orthogonal?

c) Consider the Kerr geometry. Can a hyper-surface orthogonal Killing field be constructed? Consider a linear combination of the Killing fields ∂_t and ∂_ϕ and derive conditions first for the generic metric with $ds^2 = g_{tt}dt^2 + g_{\phi\phi}d\phi^2 + g_{t\phi}(dt d\phi + d\phi dt) + \cdots$. Finally, specialize to the explicit form of the metric. (Here terms containing no differentials $d\phi$ or dt are omitted in ds^2.)

d) Identify the surface and the constants $\Omega_H^\pm$ for which $\partial_t + \Omega_H^\pm \partial_\phi$ is hyper-surface orthogonal in the Kerr geometry.