Exercise 3 *Momentum of the quantized free scalar field*

Energy H and the momentum $\mathbf{P}$ operators of a free, real scalar field ϕ are given by

$$H = \int \frac{d^3p}{(2\pi)^3} E_p \left\{ a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \frac{1}{2}[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}] \right\}, \qquad P^i = \int \frac{d^3p}{(2\pi)^3} p^i \left\{ a^\dagger_{\mathbf{p}} a_{\mathbf{p}} \right\} .$$

a) Use the expressions of the energy $P^0 = H$ and the momentum operators P^i in terms of annihilation operators in order to derive the relations,

$$H^n a_{\mathbf{p}} = a_{\mathbf{p}}(H - E_p)^n , \quad (P^i)^n a_{\mathbf{p}} = a_{\mathbf{p}}(P^i - p^i)^n .$$

Derive also the analogous relations for the conjugate creation operators $a^\dagger_{\mathbf{p}}$.

b) Use the relations of the previous question in order to show that the field operators in distinct space-time points can be related,

$$e^{iPx} a_{\mathbf{p}} e^{-iPx} = a_{\mathbf{p}} e^{-ipx} , \quad e^{iPx} a^\dagger_{\mathbf{p}} e^{-iPx} = a^\dagger_{\mathbf{p}} e^{ipx} .$$

c) Use the above relations and the explicit mode expansion of the operator ϕ to argue that,

$$\phi(\mathbf{x}, t) = e^{iPx} \phi(0, 0) e^{-iPx}.$$

Exercise 4 *The free complex scalar field – quantization*

Consider the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is

$$S = \int d^4x (\partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^*).$$

It is easiest to analyze this theory by considering ϕ and ϕ^*, rather than real and imaginary parts of ϕ, as the basic dynamical variables.

a) Find the conjugate momenta π of ϕ and π^* of ϕ^* and give the canonical commutation relations of the associated field operators $\phi, \phi^\dagger, \pi$ and $\pi^\dagger$. (It will be important below to stick to this naming convention for the conjugate momenta.) Show that the Hamiltonian of this field theory is given by,

$$H = \int d^3x (\pi \pi^\dagger + \partial_i \phi \partial_i \phi^\dagger + m^2 \phi \phi^\dagger).$$

b) Compute the Heisenberg equation of motion for ϕ and $\pi^\dagger$ and show that they lead to the Klein-Gordon equation for the operator ϕ.

Exercise 5 *The free complex scalar field – key operators*

Consider the field operator ϕ of the free, complex Klein-Gordon field describing a spin-0 boson of mass m and electric charge q. The plane-wave expansion of ϕ is given by

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_{\mathbf{p}}\, \mathrm{e}^{-ipx} + b^\dagger_{\mathbf{p}}\, \mathrm{e}^{ipx}\right],$$

where $a_{\mathbf{p}}$, $a^\dagger_{\mathbf{p}}$ are the annihilation and creation operators for the particle, respectively, and likewise $b_{\mathbf{p}}$, $b^\dagger_{\mathbf{p}}$ for the corresponding anti particle. The frequencies of the modes is given by $E_p = \sqrt{\mathbf{p}^2 + m^2}$.

a) Give the mode expansions of the operators $\phi^\dagger$, π and $\pi^\dagger$. To this end use the definition of the conjugate momenta $\pi = \dot{\phi}^\dagger$ and $\pi^\dagger = \dot{\phi}$.

b) Express the charge operator

$$Q = \int d^3x\, \frac{i}{2}\, \left[\phi\pi - \pi^\dagger\phi^\dagger\right]$$

in terms of annihilation and creation operators.

c) Express the Hamiltonian H and momentum operators P^i

$$H = \int d^3x(\pi\pi^\dagger + \partial_i\phi\partial_i\phi^\dagger + m^2\phi\phi^\dagger)\,,$$

$$P^i = -\int d^3x(\pi\partial_i\phi + \pi^\dagger\partial_i\phi^\dagger)$$

in terms of creation and annihilation operators.

d) Discuss the single-particle states and show that the theory contains two sets of particles of mass m. Evaluate the charge of the particles of each type.