

1 Evolution equations in Mellin moments

- (a) Show that any non-singlet combination of quark distributions, like $V(x, t) = q_i(x, t) - q_j(x, t)$, fulfills (at leading order):

$$t \frac{\partial}{\partial t} V(j, t) = \frac{\alpha_s(t)}{2\pi} \gamma_{qq}^{(0)}(j) V(j, t), \quad (1)$$

with $V(j, t)$ the Mellin transform of $V(x, t)$ and $\gamma_{qq}^{(0)}(j)$ the so called anomalous dimension.

- (b) Compute $\gamma_{\{nm\}}^{(0)}(j)$ for all parton combinations $\{nm\}$, with $j > 1$ an integer.
 (c) The singlet distribution $\Sigma(x, t)$ is defined according to:

$$\Sigma(x, t) = \sum_i (q_i(x, t) + \bar{q}_i(x, t)). \quad (2)$$

Show that for the second moment ($j = 2$), Σ and the gluon distribution evolve according to the matrix equation:

$$t \frac{\partial}{\partial t} \begin{pmatrix} \Sigma(2, t) \\ g(2, t) \end{pmatrix} = \frac{\alpha_s(t)}{2\pi} \begin{pmatrix} -\frac{4}{3}C_F & \frac{1}{3}n_f \\ \frac{4}{3}C_F & -\frac{1}{3}n_f \end{pmatrix} \begin{pmatrix} \Sigma(2, t) \\ g(2, t) \end{pmatrix}. \quad (3)$$

- (d) Using the last expression, show that the momentum carried by all quarks and gluons is independent of t .

2 Backward evolution kernel

Originally, in 1985, T. Sjöstrand showed that backward evolution in a parton shower could be carried employing an expression for the probability of evolving backwards from (t_2, x) to (t_1, x) without resolvable branching:

$$\Pi(t_1, t_2; x) = \exp \left(- \int_{t_1}^{t_2} \frac{dt}{t} \int \frac{dz}{z} \frac{\alpha_s}{2\pi} \hat{P}(z) \frac{f(x/z, t)}{f(x, t)} \right). \quad (4)$$

Show that this expression is equivalent to the one obtained during the lectures.

3 Gluon PDF at small x

For very small x it is possible to solve the Altarelli-Parisi (AP) equations analytically, using some physically motivated approximations. This exercise follows one from the book of Peskin and Schroeder.

- (a) Show that the Q^2 dependence of the right-hand side of the AP equations can be expressed as differential equations in the variable $\xi = \log \log (Q^2/\Lambda^2)$ with Λ the characteristic scale of QCD. The running of $\alpha_s(Q^2)$ is taken at leading order.

(b) For small x we will assume that the gluon PDF diverges as x^{-1} , and we will write:

$$dx f_g(x) \sim \frac{dx}{x} . \quad (5)$$

Considering the two approximations (1) the terms involving the gluon PDF completely dominate the right hand side of the AP equations; and (2) the function $\tilde{g}(x, Q^2) = x f_g(x, Q^2)$ is a slowly varying function of x , show the approximate relation for small x :

$$\frac{\partial^2}{\partial w \partial \xi} \tilde{g}(x, \xi) = \frac{12}{\beta_0} \tilde{g}(x, \xi) , \quad (6)$$

where $w = \log(1/x)$ and $\beta_0 = 11 - \frac{2}{3}n_f$.

(c) Show that if $w\xi \gg 1$, the previous equation has the approximate solution:

$$\tilde{g}(x, \xi) = K(Q^2) \exp \left(\left[\frac{48}{\beta_0} w(\xi - \xi_0) \right]^{\frac{1}{2}} \right) , \quad (7)$$

with $K(Q^2)$ an initial condition.