Quantum Field Theory II – WS16 Due 28.10.2016

Exercises – Set 1

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Gauge Theories – First Set of Exercises

1 Free actions

Consider the following Lagrangians,

$$\mathcal{L}_{qed} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - M)\psi - eQA_\mu\bar{\psi}\gamma^\mu\psi\,, \tag{1}$$

$$\mathcal{L}_s = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\,, \tag{2}$$

$$\mathcal{L}_{cs} = \partial_\mu\phi_a^\dagger\partial^\mu\phi_b\delta^{ab} - V(\phi^a)\,. \tag{3}$$

(a) Justify the sign and normalisation of the kinetic terms. What determines the sign of the potential term?

(b) Show that $\mathcal{L}_{qed}$ is gauge invariant.

(c) Compute the equations of motion from requiring that the functional derivative of the respective actions,

$$\frac{\delta}{\delta\psi_l(x)}I(\psi_l) = 0\,, \quad I(\psi_l) = \int d^4x\mathcal{L}(\psi_l, \partial_\mu\psi_l)\,, \tag{4}$$

vanishes. It is OK to compute this for a generic Lagrangian and then insert the explicit form of the Lagrangians.

(d) Compute the canonical momenta.

2 Non-abelian field strength tensor

Compute the field strength tensor $F^a_{\mu\nu}$ from the commutator of covariant derivatives acting on a field ψ_l,

$$([D_\mu, D_\nu]\psi)_l = -iF^a_{\mu\nu}(t_a)_l^m\psi_m\,. \tag{5}$$

Use the gauge transformations of the gauge potential,

$$\delta A^a_\mu = \partial_\mu\epsilon^a + C^a_{cb}\epsilon^b A^c_\mu\,, \tag{6}$$

to obtain the transformation behavior of the field strength tensor $\delta F^a_{\mu\nu}$.

3 Group properties

Given are the Lie-algebra generators t_a of $SU(3)$ in the form of the Gell-Mann matrices λ_a,

$$t_a = \frac{1}{2}\lambda_a\,, \tag{7}$$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{8}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{9}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{10}$$

(a) Compute the structure constants C^a_{bc} from the commutators,

$$[t_a, t_b] = iC^c_{ab}t_c\,. \tag{11}$$

Feel free to employ a computer algebra system (e.g. FORM, Maple, Mathematica, etc) for this task.

(b) Write out the Jacobi identity in terms of the structure constants. Show that the adjoint generators $(t^A_i)^a_b$,

$$(t^A_i)^a_b = -iC^a_{ib}\,, \tag{12}$$

fulfill the commutator relation of the Lie algebra.

(c) Compute the functions C_A, C_F, T_A and T_F.

4 The Haar Measure for $SU(2)$

Compute the Haar measure of the group $SU(2)$ with the following instructions. The group is conveniently parametrized using the three angles θ, ϕ and ψ and Lie algebra generators proportional to the Pauli matrices,

$$\alpha^a = \theta\{\cos(\psi)\sin(\phi), \sin(\psi)\sin(\phi), \cos(\phi)\}\,, \tag{13}$$

$$\tau_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{14}$$

With this, any element of the group g can be expressed as:

$$\begin{aligned} g(\theta, \phi, \psi) &= \exp(i\alpha^a\tau_a) \\ &= \begin{pmatrix} \cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2})\cos(\phi) + & i\sin(\frac{\theta}{2})\sin(\phi)\exp(-i\psi) \\ i\sin(\frac{\theta}{2})\sin(\phi)\exp(i\psi) & \cos(\frac{\theta}{2}) - i\sin(\frac{\theta}{2})\cos(\phi) \end{pmatrix}, \end{aligned} \tag{15}$$

with $0 \le \theta \le 2\pi$, $0 \le \phi < \pi$ and $0 \le \psi \le 2\pi$.

(a) Verify the relation,

$$\mathrm{Tr}(\tau_a\tau_b) = T_F\delta_{ab}\,, \quad T_F = \frac{1}{2}\,. \tag{16}$$

(b) Compute the tangent vectors at the unit point corresponding to the directions θ, ϕ and ψ. Give the generators t_θ, t_ψ and t_ϕ.

(c) The tangent vectors at a generic point of the group are translated back to the unit point using,

$$g^{-1}\frac{\partial}{\partial x^a}g = \sum i t_a M^{ab}\,, \quad x^a = \{\theta, \psi, \phi\}\,, \tag{17}$$

defining the matrix $M(\theta, \psi, \phi)$.

(d) Compute the measure,

$$d\mu(\theta, \psi, \phi) = \frac{1}{V}\det(M) d\theta d\psi d\phi\,. \tag{18}$$