Quantum Field Theory II – WS16 Due 4.11.2016

Exercises – Set 2

Profs. H. Ita and F. Febres Cordero
Universität Freiburg

Effective actions – Second Set of Exercises

1 Grassmann integration

The left and right eigenstates of the fermionic coordinates Q_a are given by,

$$|q\rangle = \exp[-i\sum_a P_a q_a]|0\rangle\,, \tag{1}$$

$$\langle q| = \langle 0|\prod_b Q_b \exp[-i\sum_a q_a P_a]\,. \tag{2}$$

(a) Show that the above states do in fact fulfill the eigenvalue equations,

$$Q_a|q\rangle = q_a|q\rangle\,, \quad \langle q|Q_a = \langle q|q_a\,. \tag{3}$$

(b) Show that the scalar product of two states with distinct eigenvalues takes the following form,

$$\langle q'|q\rangle = \prod_a (q_a - q'_a)\,. \tag{4}$$

(c) Show that the function,

$$\tilde{\delta}(q-q') := \prod_a (q_a - q'_a)\,, \tag{5}$$

acts as a Grassmannian δ-function fulfilling the relation,

$$f(q') = \int \tilde{\prod_b} dq_b \prod_a (q_a - q'_a) f(q)\,. \tag{6}$$

Remark: use the identity $f(q) = f(q' + (q - q'))$ to evaluate the surviving terms in the integrand. Do you obtain the same result when using $\tilde{\delta}(q' - q)$?

2 Gaussian integrals

Consider the (bosonic) Gaussian integral,

$$Z = \int \prod_c dq_c \exp\left[-I(q)\right]\,, \quad I = \frac{1}{2} D_{ab} q_a q_b + J_a q_a + M\,, \tag{7}$$

with D a positive-definite symmetric matrix, and J_a and M real numbers. Show that this integral is given in terms of the stationary point (denoted by $\bar{q}_a$) of the function $I(q)$:

$$Z = \left(\det\left(\frac{D}{2\pi}\right)\right)^{-1/2} \exp\left[-I(\bar{q})\right]\,. \tag{8}$$

Remark: you will have to compute the Gaussian integral. To this end shift the integration variables in order to write $I(q)$ as a perfect square. Use the fact that the matrix D being symmetric and real, can be diagonalised by $(L^T D L)_{ab} = \delta_{ab} d_a$ with $|\det(L)| = 1$.

3 Photon Propagator

Consider the quadratic part of the gauge-fixed Yang-Mills action,

$$I_{0A} = -\int d^4x \left[\frac{1}{4}\left(\partial_\nu A_{a\mu} - \partial_\mu A_{a\nu}\right)\left(\partial^\nu A^{a\mu} - \partial^\mu A^{a\nu}\right) + \frac{1}{2\xi}\left(\partial_\mu A^{\mu a}\partial^\nu A_{\nu a}\right) + \epsilon \quad \text{terms}\right]. \tag{9}$$

(a) Compute the field operator $\mathcal{D}^{\mu\nu,ab}$ using functional derivatives,

$$\mathcal{D}^{\mu\nu,ab}(x,x') := \frac{\delta}{\delta A_{a\mu}(x)}\frac{\delta}{\delta A_{b\nu}(x')} I_{0A}\,. \tag{10}$$

(b) Give the Fourier transform of the field operator.

(c) Compute the propagator $\Delta^{\mu\nu,ab}(x,y)$,

$$\int d^4x' \mathcal{D}^{\mu\nu,ab}(x,x')\Delta_{\nu\rho,bc}(x',y) = \delta^4(x-y)\delta^\mu_\rho\delta^a_c\,. \tag{11}$$