Quantum Field Theory II – WS16 Due 27.1.2017

Exercises – Set 11

Profs. H. Ita and F. Febres Cordero
Universität Freiburg

1 Evolution equations in Mellin moments

(a) Show that any non-singlet combination of quark distributions, like $V(x,t) = q_i(x,t) - q_j(x,t)$, fulfills (at leading order):

$$t\frac{\partial}{\partial t}V(j,t) = \frac{\alpha_s(t)}{2\pi}\gamma^{(0)}_{qq}(j)V(j,t)\ , \tag{1}$$

with $V(j,t)$ the Mellin transform of $V(x,t)$ and $\gamma^{(0)}_{qq}(j)$ the so called anomalous dimension.

(b) Compute $\gamma^{(0)}_{\{nm\}}(j)$ for all parton combinations $\{nm\}$, with $j > 1$ an integer.

(c) The singlet distribution $\Sigma(x,t)$ is defined according to:

$$\Sigma(x,t) = \sum_i (q_i(x,t) + \bar{q}_i(x,t))\ . \tag{2}$$

Show that for the second moment ($j = 2$), Σ and the gluon distribution evolve according to the matrix equation:

$$t\frac{\partial}{\partial t}\begin{pmatrix}\Sigma(2,t)\\ g(2,t)\end{pmatrix} = \frac{\alpha_s(t)}{2\pi}\begin{pmatrix}-\frac{4}{3}C_F & \frac{1}{3}n_f\\ \frac{4}{3}C_F & -\frac{1}{3}n_f\end{pmatrix}\begin{pmatrix}\Sigma(2,t)\\ g(2,t)\end{pmatrix}\ . \tag{3}$$

(d) Using the last expression, show that the momentum carried by all quarks and gluons is independent of t.

2 Backward evolution kernel

Originally, in 1985, T. Söjstrand showed that backward evolution in a parton shower could be carried employing an expression for the probability of evolving backwards from (t_2, x) to (t_1, x) without resolvable branching:

$$\Pi(t_1,t_2;x) = \exp\left(-\int_{t_1}^{t_2}\frac{dt}{t}\int\frac{dz}{z}\frac{\alpha_s}{2\pi}\hat{P}(z)\frac{f(x/z,t)}{f(x,t)}\right)\ . \tag{4}$$

Show that this expression is equivalent to the one obtained during the lectures.

3 Gluon PDF at small x

For very small x it is possible to solve the Altarelli-Parisi (AP) equations analytically, using some physically motivated approximations. This exercise follows one from the book of Peskin and Schroeder.

(a) Show that the Q^2 dependence of the right-hand side of the AP equations can be expressed as differential equations in the variable $\xi = \log\log\left(Q^2/\Lambda^2\right)$ with Λ the characteristic scale of QCD. The running of $\alpha_s(Q^2)$ is taken at leading order.

(b) For small x we will assume that the gluon PDF diverges as x^{-1}, and we will write:

$$dx f_g(x) \sim \frac{dx}{x} \ . \tag{5}$$

Considering the two approximations (1) the terms involving the gluon PDF completely dominate the right hand side of the AP equations; and (2) the function $\tilde{g}(x, Q^2) = x f_g(x, Q^2)$ is a slowly varying function of x, show the approximate relation for small x:

$$\frac{\partial^2}{\partial w \partial \xi} \tilde{g}(x, Q^2) = \frac{12}{\beta_0} \tilde{g}(x, \xi) \ , \tag{6}$$

where $w = \log(1/x)$ and $\beta_0 = 11 - \frac{2}{3} n_f$.

(c) Show that if $w\xi \gg 1$, the previous equation has the approximate solution:

$$\tilde{g}(x, Q^2) = K(Q^2) \exp\left(\left[\frac{48}{\beta_0} w(\xi - \xi_0) \right]^{\frac{1}{2}} \right) \ . \tag{7}$$