Exercises for QFTI SS 2017

Exercise 25 *Properties of the Dirac equation* (3 points)

a) Show the relation $\sigma^\mu p_\mu \bar{\sigma}^\nu p_\nu = m^2$.

b) Show that the Dirac spinors $u_s(p)e^{-ixp}$ and $v_s(p)e^{ixp}$ with,

$$u_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix}, \quad v_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta_s \\ -\sqrt{p\cdot\bar{\sigma}}\eta_s \end{pmatrix}, \tag{1}$$

solve the Dirac equation.

c) Show that the matrix valued function S_F,

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(p_\mu\gamma^\mu + m)}{p^2 - m^2 + i\varepsilon} e^{-ip(x-y)} \tag{2}$$

is a Green's function of the Dirac equation,

$$(i\gamma^\mu p_\mu - m)S_R(x-y) = i\delta^{(4)}(x-y)\mathbf{1}_{4x4}\,. \tag{3}$$

Exercise 26 *Anti-commutators and supersymmetry* (6 points)

Consider the complex scalar theory,

$$\mathcal{L}_\phi = \partial_\mu\phi\partial^\mu\phi^* - m^2\phi\phi^*\,, \tag{4}$$

and that of Dirac fermions

$$\mathcal{L}_\psi = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi\,. \tag{5}$$

a) Show that the energy momentum tensors of the two theories are given by,

$$T^\phi_{\mu\nu} = \partial_\mu\phi^*\partial_\nu\phi + \partial_\mu\phi\partial_\nu\phi^* - g_{\mu\nu}\mathcal{L}_\phi\,, \tag{6}$$

$$T^\psi_{\mu\nu} = i\bar{\psi}\gamma_\mu\partial_\nu\psi - g_{\mu\nu}\mathcal{L}_\psi\,. \tag{7}$$

b) Show the relations $\bar{u}_s(p)\gamma^0 u_{s'}(p) = 2E_p\delta_{ss'}$ and $u^\dagger_s(p)v_{s'}(-p) = v^\dagger_s(p)u_{s'}(-p) = 0$. Compute $\bar{v}_s(p)\gamma^0 v_{s'}(p)$.

c) Express the Hamiltonians of the two theories in terms of creation and annihilation operators. Do not use a normal ordering prescription and keep the vacuum-energy terms $\delta^{(3)}(0)$. What goes wrong if you use the wrong commutation prescription for the respective theories? (3 points)

d) In a field theory with n_ϕ and n_ψ Dirac fermions, how many fields do you have to combine so that the vacuum energy cancels?

Exercise 27 *Gauge fixing* **Bonus exercise (3 points)**

Consider the field theory of the electromagnetic potential A_μ,

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \lambda (\partial_\mu A^\mu)^2 \,, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{8}$$

and λ a non-vanishing constant parameter.

a) Compute the matrix representing the field operator in momentum space.

b) Show that the field operator is invertible for generic values of λ. Is it invertible for $\lambda = 0$?

c) Give the definition of a Green's function. Compute the inverse of the field operator and give the Green's function.