Exercises for QFTI **SS 2017**

Exercise 28 *Properties of relativistic field equations* (3 points)

Consider the transformation behaviour of the classical scalar, vector and spinor fields under Lorentz transformations,

$$\begin{aligned} \phi(x) &\rightarrow \phi'(x) = \phi(\Lambda^{-1}x)\,, & (1)\\ A_\mu(x) &\rightarrow A'_\mu(x) = \Lambda_\mu{}^\nu A_\nu(\Lambda^{-1}x) & (2)\\ \psi(x) &\rightarrow \psi'(x) = \Lambda_{1/2}\,\psi(\Lambda^{-1}x)\,. & (3) \end{aligned}$$

a) Show that the transformed fields solve their respective field equations,

$$\begin{aligned} &\partial_\mu\partial^\mu A'_\nu(x) - \partial_\nu\partial_\mu A'^\mu(x) = 0\,, & (4)\\ &(\partial_\mu\partial^\mu + m^2)\phi'(x) = 0\,, & (5)\\ &(i\gamma^\mu\partial_\mu - m)\psi'(x) = 0\,. & (6) \end{aligned}$$

b) Field configurations $A_\mu(x)$, $\psi(x)$ and $\phi(x)$ that do not solve the field equations (off-shell fields) are reducible under the Poincare group. Can you identify irreducible representations of the Poincare group which are embedded in the fields. E.g. solutions of the field equations as well as constant fields are such irreducible components. Give further classes of irreducible components that are embedded in the off-shell fields.

Exercise 29 *Unitary representation of the Lorentz group and spinor operators* (4 points)

a) The solution of the Dirac equation, $u_s(p)e^{-ip\cdot x}$,

$$u_s(p) = \begin{pmatrix} \sqrt{\sigma\cdot p}\,\xi_s \\ \sqrt{\bar\sigma\cdot p}\,\xi_s \end{pmatrix} \qquad (7)$$

can be obtained from Lorentz transformations of the rest frame ($k^\mu = (m,0,0,0)$) spinors,

$$u_s(k) = \sqrt{m}\begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}\,. \qquad (8)$$

Which Lorentz transformations, $p = \Lambda(p)\cdot k$, accomplish this?

b) Assume the transformation behaviour

$$U(\Lambda)a_p^{i\dagger}U^\dagger(\Lambda) = \sqrt{\frac{E_{\Lambda p}}{E_p}}\, Q_j{}^i(\Lambda, p)\, a_{\Lambda p}^{j\dagger} \tag{9}$$

$$\Lambda_{1/2}u_i(p) = Q^j{}_i(\Lambda, p)u_j(\Lambda p)\,, \tag{10}$$

with the matrices $Q(\Lambda, p)$ being unitary little group transformations.

Show that the operator,

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}a^s(p)u_s(p)\, e^{-ix\cdot p} + \cdots \tag{11}$$

transforms in the following way,

$$U(\Lambda)\psi(x)U^\dagger(\Lambda) = \Lambda_{1/2}^{-1}\psi(\Lambda x)\,. \tag{12}$$

c) How does the expectation value of the field operator transform under Lorentz transformations of a generic state $|\chi\rangle$ of the theory?

Exercise 30 *Coulomb potential* (3 points)

Consider the scattering of two electrons and one electron and a positron, respectively. The leading contribution to the matrix element of the electron scattering arise from the correlation function,

$$i\mathcal{M} \sim \langle 0|a_{p'}^{s'}a_{k'}^{s'}\, T[(i\int dx^4\mathcal{L}_{int})^2]\, a_p^{s\dagger}a_k^{s\dagger}|0\rangle\,, \tag{13}$$

$$\mathcal{L}_{int} = -e\bar{\psi}\gamma^\mu A_\mu\psi\,. \tag{14}$$

a) Evaluate the correlator in the limit of vanishing electron momentum and draw the associated Feynman diagram.

b) Which correlator describes electron-positron scattering? What is the value of the leading contribution in the small-momentum limit?

c) Identify the Coulomb potential and interpret the result. What difference between the two cases is expected?