Exercises for QFTI SS 2017

Exercise 4 *Quantum-field commutators* (2 points)

Given that one knows the quantum mechanics of the harmonic oscillator, one can already compute with quantum fields.

Consider the quantum field of the Klein-Gordon theory,

$$\phi(\vec{x},0) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p e^{i\vec{p}\vec{x}} + a_p^\dagger e^{-i\vec{p}\vec{x}}) \,, \tag{1}$$

with $\omega_p := \sqrt{\vec{p}^2 + m^2}$ and the commutator relations,

$$[a_p, a_k] = 0 \,, \quad [a_p^\dagger, a_k^\dagger] = 0 \,, \quad [a_p, a_k^\dagger] = (2\pi)^3 \delta^3(\vec{p} - \vec{k}) \,. \tag{2}$$

a) Compute the equal-time commutator relations, $[\phi(\vec{x},0), \phi(\vec{y},0)] = 0$.

b) Compute the equal-time commutator relations, $[\pi(\vec{x},0), \phi(\vec{y},0)] = -i\delta^3(\vec{x} - \vec{y})$ with $\pi(\vec{x},t) := \partial_t \phi(\vec{x},t)$.

Exercise 5 *Yukawa potential (S)* (3 points)

a) Calculate the equations of motion for the massive vector field A_μ from,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu - A_\mu J^\mu \,, \tag{3}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Assume that the current is conserved and derive a constraint from the equations of motion for A_μ.

b) Solve the equations of motion for a current J_μ associated to a point charge to obtain,

$$A_0(r) = \frac{e}{4\pi^2 i r} \int_{-\infty}^{\infty} \frac{k dk}{k^2 + m^2} e^{ikr} \tag{4}$$

c) Evaluate the integral with contour integration to obtain an explicit form for $A_0(r)$. Finally take the limit of $m \to 0$ to obtain the Coulomb potential.

Exercise 6 *Properties of Green's functions* (5 points)

In the following consider the Greens functions of the Klein-Gordon field equation,

$$(\partial_\mu \partial^\mu + m^2) \Pi_i(x, x') = -i\delta^4(x - x') \,. \tag{5}$$

a) Are the advanced $\Pi_A(x, x')$, retarded $\Pi_R(x, x')$ and Feynman $\Pi_F(x, x')$ Green's functions invariant under Lorentz transformations and translations?

b) Show that the retarded and advanced Green's functions are related under $\Pi_R(x, x') = \Pi_A(x', x)$.

c) Consider the differences of the above Greens functions $\Pi_i - \Pi_j$ and represent them as contour integrals.

d) Check that the functions,

$$\Pi_0(x, x') = \int \frac{d^4k}{(2\pi)^4} \delta(k^2 - m^2) \tilde{\Pi}_0(\vec{k}, \mathrm{sign}(k_0)) e^{-ik(x-x')} \,, \tag{6}$$

solve the homogeneous Klein-Gordon equation for regular functions $\tilde{\Pi}_0$.

e) The differences of the Greens functions solve the homogeneous field equations. Find the Fourier representation (see previous question) of the respective functions.