Exercises for QFTI SS 2017

Exercise 7 *Quantisation of free complex scalar field* (5 points)

Consider the Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \, . \tag{1}$$

a) Compute the Noether charge associated to the phase symmetry $\phi \to e^{-i\alpha}\phi$.

b) Compute the Noether charges of the translation symmetry, P^μ.

c) Use the form of the field operator,

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{-ixp} + b^\dagger e^{ixp} \right) , \tag{2}$$

with two types of creation and annihilation operators,

$$[a_p, a_q^\dagger] = (2\pi)^3 \delta^3(p-q) \, , \quad [b_p, b_q^\dagger] = (2\pi)^3 \delta^3(p-q) \, , \tag{3}$$

and all remaining commutators vanishing. Compute the normal ordered expressions of the operators $H = P^0, \vec{P}$ and Q. (Remark: $\phi^\dagger$ is obtained by hermitian conjugation.)

d) Give the quantum numbers of the single-particle states,

$$|p, a\rangle = (\sqrt{2\omega_p}) a_p^\dagger |0\rangle \, , \quad |p, b\rangle = (\sqrt{2\omega_p}) b_p^\dagger |0\rangle \, , \tag{4}$$

corresponding to the eigenvalues of the operators Q and P^μ. What is the scalar product of the states, $\langle p, a | q, b \rangle$.

e) Show that the field operators ϕ and $\phi^\dagger$ are equivalent to two independent scalar fields $\phi_{i=1,2}$ with $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$.

Exercise 8 *Toy Feynman rules* (2 points)

Consider the polynomial relation,

$$m \, \phi = \lambda \phi^2 + j \, . \tag{5}$$

a) Solve for ϕ and expand the solutions for small λ. Next solve the equations for applying the Green function method: What is the field operator? What corresponds to the Greens function?

b) Use the Green-function method to solve the polynomial equation to third order in the coupling λ. To this end draw the Feynman diagrams and use the Feynman rules. Compare to the results obtained earlier in (a).

Exercise 9 *Contour integrals and commutators* (3 points)

a) Compute the momentum-space representation of the commutator,

$$[\phi(x,t), \phi(y,t')] \tag{6}$$

for $t \neq t'$ for the Heisenberg operators $\phi(\vec{x},t)$ of the free scalar Klein-Gordon theory. (Remark: explicitly insert the mode expansions of the operators ϕ.)

b) Show that the commutator is a homogeneous solution to the free-field operator, $(\Box_{x,t} + m^2)[\phi(x,t), \phi(y,t')] = 0$.

c) Show the relation,

$$\theta(t-t')[\phi(x,t), \phi(y,t')] = \Pi_R(x-y, t-t')\,, \tag{7}$$

for the retarded Green function $\Pi_R(x-y, t-t')$. How does it work out that the commutator solves the homogeneous field equation?

Exercise 10 *Lorentz invariance* (2 points)

a) Show that the expression,

$$\omega_p \delta^3(p-q)\,, \quad \omega_p = \sqrt{p^2+m^2}\,, \tag{8}$$

is invariant under the boosts,

$$\Lambda = \begin{pmatrix} \cosh\theta & \sinh\theta & & \\ \sinh\theta & \cosh\theta & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{9}$$

b) Confirm that Λ is in fact a Lorentz transformation for generic values of θ. Draw the parts of the mass shell that are generated by acting with Λ on the four-momentum vectors $p_\mu^\pm = (\pm m, 0, 0, 0)$.