Solutions for QFTI Exercises Sheet 6 **SS 2017**

Exercise 15 *Probability conservation in QFT*

a) The associated symmetry transformation is the a change of phase of the wave function. I.e. if the wave function is transformed as

$$\psi(\vec{x},t) \to \psi'(\vec{x},t) = e^{-i\theta}\psi(\vec{x},t),$$

then the probability remains locally invariant as

$$|\psi(\vec{x},t)|^2 \to |\psi'(\vec{x},t)|^2 = |e^{-i\theta}\psi(\vec{x},t)|^2 = |\psi(\vec{x},t)|^2.$$

Notably, the Schrödinger equation is unchanged by the transformation, taking

$$\hat{H}\psi(\vec{x},t) = i\frac{\partial}{\partial t}\psi(\vec{x},t), \quad \hat{H} = -\frac{1}{2m}\nabla^2 + V(\vec{x}),$$

we see that transformation provides only a constant factor which we can cancel.

Further, the Schrödinger Lagrangian,

$$\mathcal{L} = \frac{1}{2m}(\nabla\psi^*)(\nabla\psi) + i\psi^*\frac{\partial}{\partial t}\psi - \psi^* V(\vec{x})\psi,$$

is also manifestly invariant under $\psi \to e^{-i\theta}\psi$, $\psi^* \to e^{i\theta}\psi^*$.

The conserved current is given by

$$\begin{aligned} J_t &= \psi^*\psi, \\ J_i &= \frac{i}{2m}(\psi^*\frac{\partial}{\partial x_i}\psi - \psi\frac{\partial}{\partial x_i}\psi^*). \end{aligned}$$

b) The momentum operator in quantum mechanics is given by $-i\frac{\partial}{\partial x_i}$. Here there is one conjugate momentum for each degree of freedom x_i. In analogy, the conjugate momentum operator, which will act on the wave functional is by analogy given by

$$-i\frac{\delta}{\delta\phi(\vec{x},t_0)}.$$

[We use the common notation $\delta/\delta\phi(\vec{x},t_0)$ for derivatives with respect ot field variables; when discretizing space this stands for $\partial/(\partial\phi_{\vec{x}}(t_0))$.] Here we have one conjugate momentum operator for each degree of freedom $\phi(\vec{x},t_0)$, which are labelled by each position $\vec{x}$.

c) To construct the Schrödinger equation we must promote the Hamiltonian to an operator, i.e.:

$$\hat{H} = \int d^3x \left(-\frac{\delta^2}{(\delta\phi(\vec{x},t_0))^2} + \partial_i\phi(\vec{x},t_0)\partial_i\phi(\vec{x},t_0) + V[\phi(\vec{x},t_0)] \right).$$

The Schrödinger equation is then given by

$$(\hat{H}[\phi] - i\partial_t)\psi(\phi,t) = 0.$$

The Lagrangian is then given by

$$\mathcal{L} = \psi^*(\phi,t)(\hat{H}[\phi] - i\partial_t)\psi(\phi,t).$$

Further, our action, for which the Schrödinger equation finds an extremum, is

$$\int dt \left[\prod_{\vec{x}} d\phi(\vec{x},t_0) \right] \mathcal{L}(\psi(\phi(\vec{x},t_0),t).$$

I.e. we must integrate over the degrees of freedom.

d) By analogy with the univariate case in part a, the conserved current is

$$J_t = \psi^*(\phi(\vec{x},t_0),t)\psi(\phi(\vec{x},t_0),t),$$

$$J_{\vec{x}} = \frac{i}{2}\left(\psi^*(\phi(\vec{x},t_0),t))\frac{\delta}{\delta\phi(\vec{x},t_0)}\psi(\phi(\vec{x},t_0),t) - \psi(\phi(\vec{x},t_0)t_i)\frac{\delta}{\delta\phi(\vec{x},t_0),t)}\psi^*(\phi(\vec{x},t_0),,t) \right).$$

As such, the conserved charge is given by

$$Q = \int \left[\prod_{\vec{x}} d\phi(\vec{x},t_0) \right] \psi^*(\phi(\vec{x},t_i))\psi(\phi(\vec{x},t_i)).$$

Exercise 16 *Contractions and diagrams* (3 points)

a) The correlators which have to be computed are:

$$\langle 0|0\rangle$$
$$\langle 0|T[\phi(x)^4]|0\rangle$$
$$\langle 0|T[\phi(x)^4\phi(y)^4]|0\rangle$$

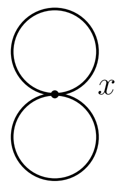
For each integral over position we get a factor of $\frac{-i\lambda}{4!}$, and further in each order we get a factor of $\frac{1}{n!}$ from the expansion of the exponential. This leads the prefactors to be:

$$c_{\mathrm{LO}} = 1,$$
$$c_{\mathrm{NLO}} = \frac{-i\lambda}{4!},$$
$$c_{\mathrm{NNLO}} = \left(\frac{-i\lambda}{4!}\right)^2 \frac{1}{2}.$$

b) The NLO contribution has the following set of contractions

$$\phi(x)\phi(x)\phi(x)\phi(x), \quad \phi(x)\phi(x)\phi(x)\phi(x), \quad \phi(x)\phi(x)\phi(x)\phi(x).$$

The single Feynman diagram for this case is

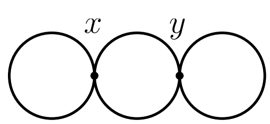


All three contractions give rise to the same term, so the prefactor of the diagram is

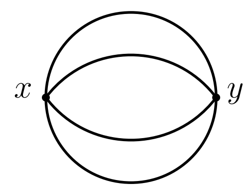
$$3\frac{-i\lambda}{4!} = \frac{-i\lambda}{8}.$$

This implies a symmetry factor of 8. This corresponds to the flips of each loop and the exchange symmetry between the two bubbles.

c) The two connected diagrams for the NNLO contributions are given by



and



We shall count the number of contractions contributing to each diagram by counting the number of ways to form each contraction, making sure not to overcount. We start with the first diagram. As there is an xx contraction, we note that we can form this in $\binom{4}{2}$ ways. Similarly, there is a yy contraction which can be chosen in $\binom{4}{2}$ ways. Next we wish to make two xy contractions. We can see by inspection in this case that there are only two, so the numerator is

$$\binom{4}{2} \cdot \binom{4}{2} \cdot 2 = 72.$$

Let us count in another way to be sure. We try to first make the two xy connections. There are 4×4 ways to make the first contraction. If we make the second contraction we have to be careful, naively there are 3×3, but this double counts as in this way we make the same two contractions in two different orders, so we have to divide by two. This gives us

$$4^2 \cdot 3^2/2 = 72.$$

We are now confident that the numeric prefactor is $\frac{72}{2 \cdot 4! \cdot 4!} = \frac{1}{16}$. This symmetry factor of 16 can be understood as the three possible flips of the bubbles, combined with the $x \leftrightarrow y$ symmetry, i.e.

$$2^3 \cdot 2 = 16.$$

For the second diagram we have to form 4 xy contractions. We can use the same logic as before, however now we have 4! different orderings by which we overcount

$$\frac{4^2 \cdot 3^2 \cdot 2^2 \cdot 1^2}{4!} = 4!.$$

This leads to a numeric prefactor of

$$\frac{1}{4! \cdot 2}.$$

The symmetry factor can be understood as a 4! coming from the exchange of propagators (note that this is actually the overcounting factor we just had to divide by) and a factor of 2 coming from the exponential.