Exercise 17 *Some properties of Lorentz transformations* (3 points)

Lorentz transformations, which transform a four-vector $a^\mu = (a^0, \vec{a})$ to $a'^\mu = \Lambda^\mu{}_\nu a^\nu$, are given by 4×4 matrices Λ that leave the metric tensor $g^{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$ invariant, i.e. $g^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g^{\alpha\beta}$. In the following we consider the "proper orthochroneous Lorentz group" $L_+^\uparrow$ that comprises all such Λ with the two constraints that $\det\Lambda = +1$ and $\Lambda^0{}_0 > 0$. The group $L_+^\uparrow$ consists of all rotations in space and "boosts", which relate two frames of reference with a non-vanishing relative velocity.

a) A boost with relative velocity $\vec{v} = (v^1, v^2, v^3)$ is described by the Λ matrix of the form

$$L(v^1,v^2,v^3)^\mu{}_\nu = \begin{pmatrix} \gamma & -v^1\gamma & -v^2\gamma & -v^3\gamma \\ -v^1\gamma & 1+\frac{v^1v^1}{\vec{v}^2}(\gamma-1) & \frac{v^1v^2}{\vec{v}^2}(\gamma-1) & \frac{v^1v^3}{\vec{v}^2}(\gamma-1) \\ -v^2\gamma & \frac{v^2v^1}{\vec{v}^2}(\gamma-1) & 1+\frac{v^2v^2}{\vec{v}^2}(\gamma-1) & \frac{v^2v^3}{\vec{v}^2}(\gamma-1) \\ -v^3\gamma & \frac{v^3v^1}{\vec{v}^2}(\gamma-1) & \frac{v^3v^2}{\vec{v}^2}(\gamma-1) & 1+\frac{v^3v^3}{\vec{v}^2}(\gamma-1) \end{pmatrix},$$

where $\gamma = 1/\sqrt{1-\vec{v}^2}$. Calculate the boosted components x'^μ for the four-vectors $x_{||}^\mu = (x^0, r\vec{e})$ and $x_\perp^\mu = (x^0, r\vec{e}_\perp)$ whose directions in space are parallel and perpendicular to the direction $\vec{e} = \vec{v}/|\vec{v}|$ of the relative velocity, respectively, i.e. $\vec{e}_\perp \cdot \vec{e} = 0$.

b) Calculate $W = L_2(0,-v^2,0)L_1(-v^1,0,0)L_2(0,v^2,0)L_1(v^1,0,0)$ for small velocities v^k and keep terms up to quadratic order in products of components v^k. What kind of transformation is described by W?

Hint: Expand the boost matrices to quadratic order in the velocities and show that they can be written in the form $L_i = \mathbf{1} - \frac{v_i}{c}K_i + \frac{1}{2}\left(\frac{v_i}{c}\right)^2 K_i^2 + \mathcal{O}\left(\frac{v_i}{c}\right)^3$ with four-by-four matrices K_i.

c) Show that the totally antisymmetric tensor

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } (\mu\nu\rho\sigma) = \text{even permutation of } (0123), \\ -1 & \text{if } (\mu\nu\rho\sigma) = \text{odd permutation of } (0123), \\ 0 & \text{otherwise.} \end{cases}$$

is an invariant tensor under all $\Lambda \in L_+^\uparrow$, i.e. $\epsilon'^{\mu\nu\rho\sigma} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\rho{}_\gamma \Lambda^\sigma{}_\delta \epsilon^{\alpha\beta\gamma\delta}$.

Exercise 18 *Commutator relations of Lorentz generators* (3 points)

Consider the defining representation fo the Lorentz group given by the rotations $\Lambda(\theta_i)$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta_x & \sin\theta_x \\ & & -\sin\theta_x & \cos\theta_x \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \cos\theta_y & & -\sin\theta_y \\ & & 1 & \\ & \sin\theta_y & & \cos\theta_y \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & \cos\theta_z & \sin\theta_z & \\ & -\sin\theta_z & \cos\theta_z & \\ & & & 1 \end{pmatrix},$$

and the boosts $\Lambda(\beta_i)$,

$$\begin{pmatrix} \cosh\beta_x & \sinh\beta_x & & \\ \sinh\beta_x & \cosh\beta_x & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \cosh\beta_y & & \sinh\beta_y & \\ & 1 & & \\ \sinh\beta_y & & \cosh\beta_y & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \cosh\beta_z & & & \sinh\beta_z \\ & 1 & & \\ & & 1 & \\ \sinh\beta_z & & & \cosh\beta_z \end{pmatrix}.$$

a) Compute the generators J_i and K_i of the Loranz transformations, which are given by the linear term of the Taylor expansions of the rotations and boosts respectively,

$$\Lambda(\theta_i) = 1 + iJ_i\theta_i + \mathcal{O}(\theta_i^2), \quad \Lambda(\beta_i) = 1 + iK_i\beta_i + \mathcal{O}(\beta_i^2). \tag{1}$$

b) Verify the commutator relations,

$$[J_i, J_j] = i\varepsilon_{ijk}J_k, \quad [J_i, K_j] = i\varepsilon_{ijk}K_k, \quad [K_i, K_j] = -i\varepsilon_{ijk}J_k. \tag{2}$$

c) Verify that the commutators of the linear combinations,

$$J_i^\pm = \frac{1}{2}(J_i \pm iK_i), \tag{3}$$

decouple into two sets J_i^+ and J_i^- which commute with one another,

$$[J_i^\pm, J_j^\pm] = i\varepsilon_{ijk}J_k^\pm, \quad [J_i^+, J_j^-] = 0. \tag{4}$$