**Exercises for QFTI SS 2017**

**Exercise 19** *Parameter range of Lorentz transformations* (4 points)

Consider the left/right-handed spinor representation $(1/2, 0)$ and $(0, 1/2)$ of the Lorentz group.

a) Give matrices corresponding to the generators of the Lie algebra associated to rotations ($J_i$) and boosts ($K_i$). Verify that the commutator relations,

$$[J_i, J_j] = i\epsilon_{ijk}J_k\,, \quad [J_i, K_j] = i\epsilon_{ijk}K_k\,, \tag{1}$$
$$[K_i, K_j] = -i\epsilon_{ijk}J_k\,, \tag{2}$$

are fulfilled. Explain, why the sign of the boost generator does not matter, so that both left and right-handed representation fulfill the Lorentz algebra.

b) Compute the group element corresponding to a rotation by the angle $\theta$ that keeps the z-axis invariant in the spinor and the vector representation,

$$\Lambda_s = \exp[i\theta(J^s)_z]\,, \quad \Lambda_V = \exp[i\theta(J^V)_z]\,. \tag{3}$$

To this end, consider the power series representation of the exponential function. Evaluate the required powers of the Lorentz generators and collect the terms that multiply identical matrices. Discuss the periodicity properties of the above rotation matrices in the angle $\theta$.

**Exercise 20** *Lorentz invariant Lagrangians for spinors* (3 points)

a) Show that the kinetic terms,

$$i\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R\,, \quad \text{and} \quad i\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L\,, \tag{4}$$

including the left and right-handed spinors are invariant under infinitesimal Lorentz transformations.

b) Furthermore, show that the Lagrangian,

$$\mathcal{L} = i\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L + i\frac{m}{2}(\psi_L^\dagger\sigma_2\psi_L^* - \psi_L^T\sigma_2\psi_L)\,, \tag{5}$$

is Lorentz invariant. [1]

[1] To avoid trivially vanishing results the field components have to be assumed to be 'Grassmann valued', meaning that they anti commute; $\psi_1(x)\psi_2(x) = -\psi_2(x)\psi_1(x)$.

c) Show that the Dirac Lagrangian,

$$\mathcal{L} = i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R + i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)\,, \tag{6}$$

has a global $U(1)$ symmetry rotating the spinors by a phase and compute the Noether current and charges. The mass terms in the Lagrangian (5) break the $U(1)$ symmetry. What is the physical implication of this? Which subgroup of the $U(1)$ remains unbroken? What is the physical implication of the remaining symmetry?

**Exercise 21** *Dirac matrix identities* (4 points)

Use the gamma matrix anti-commutator $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ to show the following identities which are important for Feynman diagram calculations involving fermions,

a) $(\gamma_*)^2 = 1\,,$

b) $\gamma_\mu \not{p} \gamma^\mu = -2\not{p}\,,$

c) $\gamma_\mu \not{p}\not{q}\not{p} \gamma^\mu = -2\not{p}\not{q}\not{p}\,,$

d) $\{\gamma_*, \gamma^\mu\} = 0\,,$

e) $Tr[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu] = 4(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\beta\mu})\,,$

f) Verify that the matrices $P_{R,L}$,

$$P_{R,L} = \frac{1}{2}\left(1 \pm \gamma_*\right)\,, \quad \gamma_* = i\gamma^0\gamma^1\gamma^2\gamma^3\,, \tag{7}$$

are projectors, i.e. $P_{R,L}^2 = P_{R,L}$. What is the rank of these projectors? What is the physical interpretation of the rank?

Consider the set of $\gamma$-matrices $\gamma^{\mu=0,..,3}$ together with $\gamma^5 := i\gamma_*$. Does this form a Clifford algebra for 5-dimensional Minkowski space?