Exercises for QFTI SS 2017

Exercise 22 *Representation theory of Clifford algebra* (3 points)

Show that the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, gives the algebra of fermionic annihilation and creation operators,

$$\{b_i, b_j\} = \{b_i^+, b_j^+\} = 0\,, \quad \{b_i, b_j^+\} = \delta_{ij}\,, \tag{1}$$

for,

$$b_0 = \frac{1}{2}(\gamma^0 + \gamma^1)\,, \quad b_0^+ = \frac{1}{2}(\gamma^0 - \gamma^1)\,, \tag{2}$$

$$b_1 = \frac{i}{2}(\gamma^2 + i\gamma^3)\,, \quad b_1^+ = \frac{i}{2}(\gamma^2 - i\gamma^3)\,. \tag{3}$$

a) Construct the Fock space of the algebra starting from the vacuum $|0\rangle$ defined here as the state with the property $b_i|0\rangle = 0$. What is the dimension of the Fock space. List all the non-trivial states.

b) Assume a new vacuum state $|0'\rangle := b_0^+|0\rangle$. Which b's should now be interpreted as creation and annihilation operators, respectively?

c) Finally, assume that the dimension D of Minkowski space is $D = 2n$. Generalise the map from γ-matrices to b's. What is the dimension of the Fock space or, equivalently, the dimension of the representation of the Clifford algebra?

Exercise 23 *Clifford algebra properties* (4 points)

Consider the Weyl representation of the Clifford algebra,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\,. \tag{4}$$

a) Show that the Lorentz generators are block diagonal.

b) Prove (without resorting to an explicit representation) that the projection operators $P_{L,R}$ commute with the Lorentz generators. Next, compute the explicit form of the projectors in the Weyl representation.

c) The charge conjugation matrix for Dirac spinors is given by $C = -i\gamma_2$ with the charge conjugation operation given by $\psi^c = C\psi^*$. Give the general form of the spinor that is its own charge conjugate, $\psi^c = \psi$.

d) Given a solution ψ of the Dirac equation,

$$(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m)\psi = 0\,, \tag{5}$$

for a fixed background vector field A_μ. Show that the charge conjugate spinor ψ^c solves the Dirac equation with the sign of the charge flipped, i.e. $e \to -e$.

Exercise 24 *Neutrino masses* (4 points)

Consider the Lagrangian for a left and a right handed neutrino with Majorana and Dirac mass terms,

$$\mathcal{L} = \nu_L^\dagger \bar{\sigma}^\mu \partial_\mu \nu_L + \nu_R^\dagger \sigma^\mu \partial_\mu \nu_R + m(\nu_L^\dagger \nu_R + \nu_R^\dagger \nu_L) + iM(\nu_R^T \sigma_2 \nu_R - \nu_R^\dagger \sigma_2 \nu_R^*) \,. \tag{6}$$

The aim is to consider the mass eigenstates. These can be obtained by first rewriting all spinors in terms of left handed Weyl spinors $\chi_L := i\sigma_2 \nu_R^*$ and ν_L.

a) Show that χ_L in fact transforms like a left-handed spinor.

b) Rewrite the Lagrangian in terms of χ_L and ν_L. Furthermore, write the expression in terms of the doublet (ν_L, χ_L).

c) Given that the doublet (ν_L, χ_L) solves the equations of motion, show that it obeys a Klein-Gordon equation.

d) Give the mass eigenstates. Suppose $M >> m$. e.g. $M = 10^{10}$ GeV and $m = 100$ GeV. What are the masses of the physical particles.