Tutorial 1

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Throughout the tutorials, we will use the conventions of *The Standard Model: A Primer* (Cliff Burgess and Guy Moore). In particular, we will use a diag$(-,+,+,+)$ metric, so that $x^2 = -x_0^2 + |\vec{x}|^2$ and $p^2 = -m^2 = -p_0^2 + |\vec{p}|^2$.

1 Space-time

In this exercise, we review some important ideas about (Minkowski) space-time, light-cones, time-like and space-like curves, ...

1.1 Light-cone:

Draw the light-cone of an event situated at the point O in space-time. Then:

- Identify the past and future light-cones ;
- Draw a light-like, a time-like, the present hypersurface and a space-like curve ;
- Identify the region of space-time that can causally affect O ;
- Identify the region of space-time that can be causally affected by O ;
- Identify the region of space-time that is causally disconnected from O.

1.2 Minkwoski geometry:

Consider a Minkowski space with one temporal component and one spatial component (instead of the usual 3 spatial components). Draw the light-cone of an event at the point O in space-time. Then:

- Draw lines with the same proper-time separation from O ;
- Draw lines with the same proper-distance separation from O.

2 Creation and annihilation operators

2.1 Bosonic statistics

Let $a_{\vec{p},\alpha}$ ($a^*_{\vec{p},\alpha}$) be the annihilation (creation) operator of a bosonic state of momentum $\vec{p}$ and quantum numbers α. They act as follows:

$$\begin{aligned} a_{\vec{p},\alpha}\,|0\rangle &= 0 \\ a_{\vec{p},\alpha}\,|\vec{q},\beta\rangle &= 2E_{\vec{p}}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{\alpha,\beta}\,|0\rangle \\ a_{\vec{p},\alpha}\,|\vec{q},\beta;\vec{r},\gamma\rangle &= 2E_{\vec{p}}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{\alpha,\beta}\,|\vec{r},\gamma\rangle + 2E_{\vec{p}}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{r})\delta_{\alpha,\gamma}\,|\vec{q},\beta\rangle\,, \end{aligned}$$

and

$$
\begin{aligned}
a^*_{\vec{p},\alpha}|0\rangle &= |\vec{p},\alpha\rangle \\
a^*_{\vec{p},\alpha}|\vec{q},\beta\rangle &= |\vec{p},\alpha;\vec{q},\beta\rangle = |\vec{q},\beta;\vec{p},\alpha\rangle\,,
\end{aligned}
$$

where in the last line we used the fact that these are bosonic states. Show the following relations

$$
\begin{aligned}
[a_{\vec{p},\alpha}, a_{\vec{q},\beta}] &= \left[a^*_{\vec{p},\alpha}, a^*_{\vec{q},\beta}\right] = 0 \\
\left[a_{\vec{p},\alpha}, a^*_{\vec{q},\beta}\right] &= 2E_{\vec{p}}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{\alpha,\beta}
\end{aligned}
$$

by acting on the vacuum state $|0\rangle$, the one-particle state $\left|\vec{t},\delta\right\rangle$ and the two-particle state $\left|\vec{t},\delta;\vec{v},\omega\right\rangle$.

2.2 Fermionic statistics

Let $b_{\vec{p},\alpha}$ ($b^*_{\vec{p},\alpha}$) be the annihilation (creation) operator of a fermionic state of momentum $\vec{p}$ and quantum numbers α. They act as follows:

$$
\begin{aligned}
b_{\vec{p},\alpha}|0\rangle &= 0 \\
b_{\vec{p},\alpha}|\vec{q},\beta\rangle &= 2E_{\vec{p}}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{\alpha,\beta}|0\rangle \\
b_{\vec{p},\alpha}|\vec{q},\beta;\vec{r},\gamma\rangle &= 2E_{\vec{p}}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{\alpha,\beta}|\vec{r},\gamma\rangle - 2E_{\vec{p}}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{r})\delta_{\alpha,\gamma}|\vec{q},\beta\rangle\,,
\end{aligned}
$$

and

$$
\begin{aligned}
b^*_{\vec{p},\alpha}|0\rangle &= |\vec{p},\alpha\rangle \\
b^*_{\vec{p},\alpha}|\vec{q},\beta\rangle &= |\vec{p},\alpha;\vec{q},\beta\rangle = -|\vec{q},\beta;\vec{p},\alpha\rangle\,,
\end{aligned}
$$

where in the last line we used the fact that these are fermionic states. Show the following relations

$$
\begin{aligned}
\{b_{\vec{p},\alpha}, b_{\vec{q},\beta}\} &= \left\{b^*_{\vec{p},\alpha}, b^*_{\vec{q},\beta}\right\} = 0 \\
\left\{b_{\vec{p},\alpha}, b^*_{\vec{q},\beta}\right\} &= 2E_{\vec{p}}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{\alpha,\beta}
\end{aligned}
$$

by acting on the vacuum state $|0\rangle$, the one-particle state $\left|\vec{t},\delta\right\rangle$ and the two-particle state $\left|\vec{t},\delta;\vec{v},\omega\right\rangle$.

3 Lorentz invariant integration measure

Show that the integration measure

$$
\int \frac{d^3\vec{p}}{2E_{\vec{p}}\,(2\pi)^3} \tag{1}
$$

is Lorentz invariant. For this, show it is equal to

$$
\int \frac{d^4p}{(2\pi)^4} 2\pi\delta(p^2+m^2)\theta(p_0) \tag{2}
$$

and argue that this expressions is invariant under (proper orthochronous) Lorentz transformations.

4 Clifford algebra

Let γ^μ, with $\mu = 0,\ldots,3$, be a set of 4×4 matrices satisfying

$$
\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{3}
$$

We also define γ_5 as

$$\gamma_5 = \gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{4}$$

Finally, we set $\not{p} \equiv p_\mu \gamma^\mu$.

An explicit representation of the γ^μ matrices is

$$\gamma^0 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \qquad \gamma_k = \begin{bmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{bmatrix}, \tag{5}$$

where each entry should be understood as a 2×2 matrix, and the σ_k, with $k = 1, 2, 3$, are the usual Pauli matrices,

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{6}$$

Show the following relations, either by using the explicit representation of eq. (5), or the defining relation of the Clifford algebra, eq. (3):

- $\gamma_\mu \gamma^\mu = 4$; $\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu$; $\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\eta^{\nu\rho}$;
- $(\gamma_5)^2 = 1$;
- $\{\gamma_5, \gamma_\mu\} = 0$;
- $\mathrm{tr}\left[\not{p}\not{q}\right] = 4p \cdot q$.

Extra questions, if the rest was very easy: Show that

- $\mathrm{tr}\left[\not{p}_1\not{p}_2\not{p}_3\not{p}_4\right] = 4\left(p_1 \cdot p_2\, p_3 \cdot p_4 - p_1 \cdot p_3\, p_2 \cdot p_4 + p_1 \cdot p_4\, p_2 \cdot p_3\right)$;
- $\mathrm{tr}\left[\not{p}_1\not{p}_2 \cdots \not{p}_n\right] = 0$ for any n odd.