Tutorial 3

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Throughout the tutorials, we will use the conventions of *The Standard Model: A Primer* (Cliff Burgess and Guy Moore). In particular, we will use a diag$(-,+,+,+)$ metric, so that $x^2 = -x_0^2 + |\vec{x}|^2$ and $p^2 = -m^2 = -p_0^2 + |\vec{p}|^2$.

1 Dirac and Klein-Gordon equations

The Dirac equation, the equation of motion of a free fermionic field ψ, is given by

$$\left(\not{\partial} + m\right)\psi = 0. \tag{1}$$

We wish to show that the Dirac equation implies the Klein-Gordon equation. For this, show that by 'squaring' the Dirac equation one finds that ψ satisfies the Klein-Gordon equation

$$\left(\partial^2 - m^2\right)\psi = 0. \tag{2}$$

Interpret this observation.

2 Solution of the Dirac equation

2.1 Solution of eq. (2)

Explain why we can write

$$\psi(x) = \sum_\sigma \int \frac{d^3\vec{k}}{(2\pi)^3 2E_k}\left(u(\vec{k},\sigma)e^{ik\cdot x} + v(\vec{k},\sigma)e^{-ik\cdot x}\right). \tag{3}$$

In this expression, what are $u(\vec{k},\sigma)$ and $v(\vec{k},\sigma)$? Given that $\psi(x)$ must also satisfy the Dirac equation, what constraints does this imply for $u(\vec{k},\sigma)$ and $v(\vec{k},\sigma)$?

2.2 Solution in the rest frame

Solve the equation

$$(i\not{k} + m)u(\vec{k},\sigma) = 0 \tag{4}$$

in the rest frame of k, i.e., for $\vec{k} = 0$.

2.3 Solution in a general frame

The solution for $\vec{k} \neq 0$ can be obtained by boosting the solution obtained in the rest frame. For simplicity, we choose $\vec{k} = (0, 0, k_3)$. Show that the solution can be written as

$$u(\vec{k},\sigma) = \begin{bmatrix} \left(\sqrt{E_{\vec{k}} + m} - \sqrt{E_{\vec{k}} - m}\sigma_3\right) & 0 \\ 0 & \left(\sqrt{E_{\vec{k}} + m} + \sqrt{E_{\vec{k}} - m}\sigma_3\right) \end{bmatrix} \Xi_\sigma, \tag{5}$$

where Ξ_σ is the solution in the rest frame (the explicit representation of the γ^μ and σ_i can be found in Tutorial 1).

3 Charge conjugation

Let ψ be a Majorana spinor,

$$\psi = \begin{bmatrix} \xi \\ \epsilon\xi^* \end{bmatrix}, \qquad \text{with} \qquad \epsilon = i\sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{6}$$

The charge conjugation is

$$\psi^c \equiv C\bar{\psi}^T \tag{7}$$

where $\bar{\psi}$ is the Dirac conjugate of ψ and C is the charge conjugation matrix,

$$C = \begin{bmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{bmatrix} \tag{8}$$

Show that $\psi = \psi^c$.

4 Additional question

In the tutorial session, we will discuss how rotations act on the vector and the spinor representations. In particular, we will see that as expected if a vector is rotated by an angle of 2π it returns to its original position, but this is not true for a spinor. If you have finished all the questions above, try to determine what is the (non-zero) angle one must rotate a spinor by to bring it back to its original position.