Tutorial 4

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1 Jacobi identity and the adjoint representation of a Lie group

1.1 Jacobi identity

Assume T^i denotes non-commuting operators (like matrices). Show that

$$\sum_{\text{cyclic perm. of \{a,b,c\}}} [T^a, [T^b, T^c]] = 0\,. \tag{1}$$

This is the *Jacobi identity.*

1.2 Adjoint representation

The generators of a Lie group satisfy the commutation relation

$$[T^a, T^b] = i f^{abc} T^c\,, \tag{2}$$

where the f^{abc} are called *structure constants* and are totally antisymmetric in their indices. The vector space generated by the generators together with the commutation relation form a *Lie algebra.*

The adjoint representation is defined as the representation whose generators $(F^a)_{bc}$ are the structure constants themselves:

$$(F^a)_{bc} = -i f^{abc}\,. \tag{3}$$

To show this is a valid representation, we must verify that they obey the commutation relation, i.e., that

$$([F^a, F^b])_{de} = i f^{abc} (F^c)_{de}\,. \tag{4}$$

Show that this follows from the Jacobi identity.

2 Gauge invariance of the QED Lagrangian

2.1 Gauge invariance

The Lagrangian of Quantum Electrodynamics is

$$\mathcal{L}_{\text{QED}} = -\bar{e}(x) \left(\not{D} + m_e \right) e(x) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}\,, \tag{5}$$

where D_μ denotes the covariant derivative,

$$D_\mu = \partial_\mu + ieA_\mu(x)\,, \tag{6}$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{7}$$

Under a gauge transformation, the electron field transforms as

$$e(x) \to e'(x) = e^{-ie\omega(x)}e(x) \tag{8}$$

and the photon as

$$A_\mu(x) \to A'_\mu(x) = A_\mu + \partial_\mu\omega(x)\,. \tag{9}$$

Show that the QED Lagrangian is invariant under this transformation.

2.2 Couplings of the photon

- Which term in the Lagrangian describes the coupling of the photon with the electromagnetic current? What parameter describes the strength of this interaction?
- Does the photon couple to itself? Why?

3 Gauge invariance of the QCD Lagrangian

The QCD Lagrangian can be schematically written in a similar way to the QED Lagrangian,

$$\mathcal{L}_{\rm QED} = -\bar{q}(x)\left(\not{D} + m_q\right)q(x) - \frac{1}{4}G^\alpha_{\mu\nu}G^{\alpha\mu\nu}\,, \tag{10}$$

but because it describes a non-abelian theory it has a much richer structure. In the above expression, we have omitted a sum over quark flavours, and we have used the same symbol for the covariant derivative as in QED, despite the fact that it is now given by a different expression,

$$D_\mu = \partial_\mu - igG^\alpha_\mu(x)\frac{\lambda^\alpha}{2}\,. \tag{11}$$

Here, λ^α with $\alpha = 1, \ldots, 8$ denotes the so-called Gell-Mann matrices, the generators of the $SU(3)$ Lie group. Finally,

$$G^\alpha_{\mu\nu} = \partial_\mu G^\alpha_\nu - \partial_\nu G^\alpha_\mu + gf^\alpha_{\beta\gamma}G^\beta_\mu G^\gamma_\nu\,. \tag{12}$$

Showing that the full QCD Lagrangian is gauge invariant is a long and rather tedious exercise. Instead, we will focus on determining how the gauge field G^α_μ must transform for the fermionic term to be gauge invariant. Given the transformation rule of the quarks field,

$$q(x) \to q'(x) = \exp(ig\omega^\alpha(x)\frac{\lambda^\alpha}{2})q(x)\,, \tag{13}$$

and writing the transformation of the gauge field as

$$G^\alpha_\mu(x) \to G^{\alpha\,\prime}_\mu(x) = G^\alpha_\mu(x) + \delta G^\alpha_\mu(x)\,, \tag{14}$$

determine what δG^{α}_{μ} should be for the fermionic term of the QCD Lagrangian to be gauge invariant, i.e., for

$$\bar{q}(x)\left(\not{D}+m_q\right)q(x) \to \bar{q}'(x)\left(\not{D}+m_q\right)q'(x)\,. \tag{15}$$

NOTE: For this exercise, you will need to use

$$\left[\frac{\lambda_\alpha}{2},\frac{\lambda_\beta}{2}\right] = i f^{\gamma}_{\alpha\beta}\frac{\lambda_\gamma}{2}\,. \tag{16}$$

If you have some free time, you can check that with the δG^{α}_{μ} you have determined above, the term $G^{\alpha}_{\mu\nu}G^{\alpha\mu\nu}$ is also gauge invariant, thus showing that the full QCD Lagrangian is.

3.1 Couplings of the gluons

- Which term in the Lagrangian describes the coupling of the gluon with the quark current? What parameter describes the strength of this interaction?
- Does the gluon couple to itself? In how many ways? What extra factors appear in comparison with the coupling with the quark current? How would one recover the behaviour of abelian theories?