Tutorial 8 — Correction

Exercise 1

We start by reviewing some results about traces of gamma functions. Show that:

1) $\mathrm{tr}[\gamma^{\mu_1} \ldots \gamma^{\mu_n}] = 0$ if n is odd.
2) $\mathrm{tr}[\gamma_5 \gamma^{\mu_1} \ldots \gamma^{\mu_n}] = 0$ if n is odd.
3) $\mathrm{tr}[\gamma^\mu \gamma^\nu] = 4\eta^{\mu\nu}$
4) $\mathrm{tr}[\gamma_5 \gamma^\mu \gamma^\nu] = 0$
5) $\mathrm{tr}[\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho] = 4(\eta^{\mu\nu}\eta^{\lambda\rho} - \eta^{\mu\lambda}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\lambda})$
6) $\mathrm{tr}[\gamma_5 \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho] = 4i\epsilon^{\mu\nu\lambda\rho}$

To show some of these results, recall that the only invariant second-rank tensor is the metric. For 4) and 6), it is useful to check the symmetries of the expressions under the exchange of two of the indices.

Correction of exercise 1

For simplicity, when there is no ambiguity in calculations I will use $\gamma^\mu \to \mu$. For instance, $\mathrm{tr}[\gamma^\mu \gamma^\nu]$ will be written as $\mathrm{tr}[\mu\nu]$

1) $\mathrm{tr}[\gamma^{\mu_1} \ldots \gamma^{\mu_n}] = 0$ if n is odd.

$$\begin{aligned}
\mathrm{tr}[\mu_1 \ldots \mu_n] &= \mathrm{tr}[\gamma_5^2 \mu_1 \ldots \mu_n] \\
&= (-1)^n \mathrm{tr}[\gamma_5 \mu_1 \ldots \mu_n \gamma_5] \\
&= -\mathrm{tr}[\mu_1 \ldots \mu_n \gamma_5^2] \\
&= -\mathrm{tr}[\mu_1 \ldots \mu_n].
\end{aligned}$$

2) $\mathrm{tr}[\gamma_5 \gamma^{\mu_1} \ldots \gamma^{\mu_n}] = 0$ if n is odd.

   γ^5 is a product of 4 γ-matrices, so this is follows from the previous result.

3) $\mathrm{tr}[\gamma^\mu \gamma^\nu] = 4\eta^{\mu\nu}$

$$\mathrm{tr}[\mu\nu] = \frac{1}{2}\mathrm{tr}[\{\mu, \nu\}] = \eta^{\mu\nu}\mathrm{tr}[\mathbb{1}_4] = 4\eta^{\mu\nu}.$$

4) $\mathrm{tr}[\gamma_5 \gamma^\mu \gamma^\nu] = 0$

$$\mathrm{tr}[\gamma_5 \mu\nu] = 2\eta^{\mu\nu}\mathrm{tr}[\gamma_5] - \mathrm{tr}[\gamma_5 \nu\mu] = -\mathrm{tr}[\gamma_5 \nu\mu].$$

   $\mathrm{tr}[\gamma_5 \gamma^\mu \gamma^\nu]$ is a rank two tensor, and must thus be proportional to the metric $\eta^{\mu\nu}$. However, the metric is symmetric under $\mu \to \nu$, while $\mathrm{tr}[\gamma_5 \gamma^\mu \gamma^\nu]$ is anti-symmetric. The only way to reconcile these two observations is for the trace to vanish.

5) $\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho] = 4(\eta^{\mu\nu}\eta^{\lambda\rho} - \eta^{\mu\lambda}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\lambda})$

$$\begin{aligned}\mathrm{tr}[\mu\nu\lambda\rho] &= \frac{1}{2}\left(\mathrm{tr}[\mu\nu\lambda\rho] + \mathrm{tr}[\nu\lambda\rho\mu]\right)\\ &= \eta^{\mu\nu}\mathrm{tr}[\lambda\rho] + \frac{1}{2}\left(\mathrm{tr}[\nu\mu\lambda\rho] + \mathrm{tr}[\nu\lambda\rho\mu]\right)\\ &= 4(\eta^{\mu\nu}\eta^{\lambda\rho} + \eta^{\mu\rho}\eta^{\nu\lambda} - \eta^{\mu\lambda}\eta^{\nu\rho}).\end{aligned}$$

6) $\mathrm{tr}[\gamma_5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho] = 4i\epsilon^{\mu\nu\lambda\rho}$

As we did in point 4), we can check this tensor is totally anti-symmetric. It must thus be proportional to the Levi-Civita tensor:

$$\mathrm{tr}[\gamma_5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho] = \alpha\epsilon^{\mu\nu\lambda\rho}.$$

We can determine the value of α by setting $\mu = 0$, $\nu = 1$, $\lambda = 2$ and $\rho = 3$:

$$\mathrm{tr}[\gamma_5\gamma^0\gamma^1\gamma^2\gamma^3] = \alpha \implies i\mathrm{tr}[\gamma_5\gamma_5] = \alpha \implies \alpha = 4i.$$

Exercise 2

Using the above results, evaluate

$$\mathrm{tr}\left[\gamma^\mu(g_V + g_A\gamma_5)(m_f + i\not{q})\gamma^\nu(g_V + g_A\gamma_5)(m_f - i\not{p})\right]. \tag{1}$$

Correction of exercise 2

Set $A^{\mu\nu} = \mathrm{tr}\left[\gamma^\mu(g_V + g_A\gamma_5)(m_f + i\not{q})\gamma^\nu(g_V + g_A\gamma_5)(m_f - i\not{p})\right]$. Keeping only the terms with an even number of γ-matrices, one finds

$$\begin{aligned}A^{\mu\nu} =& g_V^2\mathrm{tr}[\mu\not{q}\nu\not{p}] + g_V^2 m_f^2\mathrm{tr}[\mu\nu] + g_A^2\mathrm{tr}[\mu\gamma_5\not{q}\nu\gamma_5\not{p}] + g_A^2 m_f^2\mathrm{tr}[\mu\gamma_5\nu\gamma_5]\\ &+ g_V g_A m_f^2\mathrm{tr}[\{\mu,\nu\}\gamma_5] - 2g_V g_A\mathrm{tr}[\gamma_5\mu\not{q}\nu\not{p}].\end{aligned}$$

Using the results above, the remaining traces are

$$\begin{aligned}&\mathrm{tr}[\mu\gamma_5\nu\gamma_5] = -\mathrm{tr}[\mu\nu] = -4\eta^{\mu\nu}\\ &\mathrm{tr}[\mu\not{q}\nu\not{p}] = \mathrm{tr}[\mu\gamma_5\not{q}\nu\gamma_5\not{p}] = 4(p^\mu q^\nu + q^\mu p^\nu - \eta^{\mu\nu}p\cdot q)\\ &\mathrm{tr}[\{\mu,\nu\}\gamma_5] = 0\\ &\mathrm{tr}[\gamma_5\mu\not{q}\nu\not{p}] = p_\alpha q_\beta\mathrm{tr}[\gamma_5\mu\beta\nu\alpha] = -4ip_\alpha q_\beta\epsilon^{\mu\nu\alpha\beta}.\end{aligned}$$

Putting everything together,

$$A^{\mu\nu} = 4(g_V^2 - g_A^2)m_f^2\eta^{\mu\nu} + 4(g_V^2 + g_A^2)(p^\mu q^\nu + q^\mu p^\nu - \eta^{\mu\nu}p\cdot q) - 8ig_V g_A p_\alpha q_\beta\epsilon^{\mu\nu\alpha\beta}.$$

Exercise 3

We will study the decay rate $\Gamma(W^- \to e^- \bar{\nu}_e)$. This decay is controlled by the lagrangian

$$\mathcal{L} = ie_W W_\mu^- \bar{e}\gamma^\mu(g_V + g_A\gamma_5)\nu + \text{h.c.}\,. \tag{2}$$

Show that

$$\langle e(p)\bar{\nu}_e(q)|\mathcal{H}|W(k)\rangle = -ie_W\epsilon_\mu(k)\bar{u}(p)\gamma^\mu(g_V + g_A\gamma_5)v(q)\,. \tag{3}$$

For $m_e \neq m_\nu \neq 0$, show that

$$\sum_{\sigma_1\sigma_2} |\epsilon_\mu(k)\bar{u}(p)\gamma^\mu(g_V + g_A\gamma_5)v(q)|^2 = 4\left[m_e m_\nu(g_V^2 - g_A^2) + (g_V^2 + g_A^2)(2\epsilon\cdot p\epsilon\cdot q - p\cdot q)\right]. \tag{4}$$

For W^- linearly polarised in the direction $\vec{r}$, working in the rest frame of W^- and setting $m_\nu = 0$, show that

$$\frac{d\Gamma}{d\cos\theta} = \frac{e_W^2 M_W}{16\pi}(g_V^2 + g_A^2)\left(1 - \cos^2\theta\left(1 - \frac{m_e^2}{M_W^2}\right)\right)\left(1 - \frac{m_e^2}{M_W^2}\right)^2, \tag{5}$$

if θ is the angle between $\vec{p}$ and $\vec{r}$. Finally, show that the unpolarised rate is

$$\Gamma(W^- \to e^-\bar{\nu}_e) = \frac{e_W^2 M_W}{12\pi}(g_V^2 + g_A^2)\left(1 - \frac{m_e^2}{M_W^2}\right)^2\left(1 + \frac{m_e^2}{2M_W^2}\right). \tag{6}$$

Correction of exercise 3

As argued in chapter 4 of the book, we start by writing

$$\mathcal{H} = -\mathcal{L} = -ie_W W_\mu^- \bar{e}\gamma^\mu(g_V + g_A\gamma_5)\nu\,.$$

We must then evaluate

$$\begin{aligned}\mathcal{M} &= \langle e(p,\sigma_1), \bar{\nu}(q,\sigma_2)|\,\mathcal{H}\,|W(k,\lambda)\rangle \\ &= -ie_W\,\langle 0|\,b^e_{p,\sigma_1}\bar{b}^\nu_{q,\sigma_2}W_\mu^-\bar{e}\gamma^\mu(g_V + g_A\gamma_5)\nu a^*_{k,\lambda}\,|0\rangle\,.\end{aligned} \tag{7}$$

Using

$$\begin{aligned}\psi(x) &= \sum_{\sigma=\pm\frac{1}{2}}\int\frac{d^3p}{2E_p(2\pi)^3}\left(u(p,\sigma)b_{p,\sigma}e^{ip\cdot x} + v(p,\sigma)\bar{b}^*_{p,\sigma}e^{-ip\cdot x}\right) \\ \bar{\psi}(x) &= \sum_{\sigma=\pm\frac{1}{2}}\int\frac{d^3p}{2E_p(2\pi)^3}\left(\bar{u}(p,\sigma)b^*_{p,\sigma}e^{-ip\cdot x} + \bar{v}(p,\sigma)\bar{b}_{p,\sigma}e^{-ip\cdot x}\right) \\ W_\mu^- &= \sum_{\lambda=-1,0,1}\int\frac{d^3p}{2E_p(2\pi)^3}\left(\epsilon_\mu(k,\lambda)a_{k,\lambda}e^{ik\cdot x} + \text{h.c.}\right)\end{aligned}$$

we see the only non-vanishing contribution in eq. (7) is the one that comes from b^* in $\bar{\psi}$ (to match b^e_{p,σ_1}), $\bar{b}^*$ in ψ (to match $\bar{b}^\nu_{q,\sigma_2}$) and $a_{k,\lambda}$ in W_μ^- (to match $a^*_{k,\lambda}$). We thus find

$$\mathcal{M} = -ie_W\epsilon_\mu(k,\lambda)\bar{u}(p,\sigma_1)\gamma^\mu(g_V + g_A\gamma_5)v(q,\sigma_2) \tag{8}$$

To evaluate the decay rate, we must first compute $|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^*$:

$$|\mathcal{M}|^2 = \sum_{\sigma_1,\sigma_2} e_W^2 \epsilon_\mu \epsilon_\nu^* \left|\bar{u}(p,\sigma_1)\gamma^\mu(g_V + g_A\gamma_5)v(q,\sigma_2)\right| \left|\bar{u}(p,\sigma_1)\gamma^\nu(g_V + g_A\gamma_5)v(q,\sigma_2)\right|^*$$

It is a simple exercise (that you should try to do yourself) to find that

$$\left|\bar{u}(p,\sigma_1)\gamma^\nu(g_V + g_A\gamma_5)v(q,\sigma_2)\right|^* = -\left|\bar{v}(q,\sigma_2)\gamma^\nu(g_V + g_A\gamma_5)u(p,\sigma_1)\right|.$$

Then,

$$\begin{aligned}
|\mathcal{M}|^2 &= -\, e_W^2 \epsilon_\mu \epsilon_\nu^* \sum_{\sigma_1,\sigma_2} \left|\bar{u}(p,\sigma_1)\gamma^\mu(g_V + g_A\gamma_5)v(q,\sigma_2)\right| \left|\bar{v}(q,\sigma_2)\gamma^\nu(g_V + g_A\gamma_5)u(p,\sigma_1)\right| \\
&= -\, e_W^2 \epsilon_\mu \epsilon_\nu^* \sum_{\sigma_1,\sigma_2} \mathrm{tr}\left[\bar{u}(p,\sigma_1)\gamma^\mu(g_V + g_A\gamma_5)v(q,\sigma_2)\bar{v}(q,\sigma_2)\gamma^\nu(g_V + g_A\gamma_5)u(p,\sigma_1)\right] \\
&= -\, e_W^2 \epsilon_\mu \epsilon_\nu^* \sum_{\sigma_1,\sigma_2} \mathrm{tr}\left[\gamma^\mu(g_V + g_A\gamma_5)v(q,\sigma_2)\bar{v}(q,\sigma_2)\gamma^\nu(g_V + g_A\gamma_5)u(p,\sigma_1)\bar{u}(p,\sigma_1)\right]
\end{aligned}$$

We then use (see eqs. (4.16) and (4.17) of the book, but you can also try to reproduce these results yourself)

$$\sum_\sigma u(p,\sigma)\bar{u}(p,\sigma) = m_e - i\not{p}\,, \qquad \sum_\sigma v(q,\sigma)\bar{v}(q,\sigma) = -m_\nu - i\not{q}\,,$$

to get

$$|\mathcal{M}|^2 = e_W^2 \mathrm{tr}\left[\not{\epsilon}(g_V + g_A\gamma_5)(m_\nu + i\not{q})\not{\epsilon}^*(g_V + g_A\gamma_5)(m_e - i\not{p})\right] .$$

To compute the remaining trace, we can use the result of exercise 2 and find

$$|\mathcal{M}|^2 = 4e_W^2 \left[(g_V^2 - g_A^2)m_e m_\nu + (g_V^2 + g_A^2)(2\epsilon\cdot p\epsilon\cdot q - p\cdot q)\right] . \tag{9}$$

We note that we are instructed to compute the decay rate in the case of a linearly polarized W^--boson, which implies $\epsilon_\nu = \epsilon_\nu^*$. This means the term proportional to the Levi-Civita tensor in the result of exercise 2 vanishes: it is the product of a symmetric and an anti-symmetric tensor.

Before continuing with the evaluation of the decay rate of this specific process, we first make an aside about the general structure of the decay of a massive particle of momentum k into two massive particles of momenta p_i and mass m_i^2. The decay rate is related to the squared matrix element by

$$d\Gamma = \frac{1}{2E_k}\,|\mathcal{M}|^2\,(2\pi)^4\delta^4(k - p_1 - p_2)\frac{d^3\vec{p}_1 d^3\vec{p}_2}{4E_{p_1}E_{p_2}(2\pi)^6}$$

We can use three of the δ-functions to integrate over one of the three-momenta, say $\vec{p}_2$. We are then left to deal with the factor

$$2\pi\delta(E_k - E_1 - E_2)\frac{d^3\vec{p}_1}{4E_{p_1}E_{p_2}(2\pi)^3} = \delta(E_k - E_1 - E_2)\frac{|p_1|^2\,d\,|p_1|\,d^2\Omega}{16\pi^2 E_{p_1}E_{p_2}}.$$

where the conditions of the three δ-functions are implicit. To make the condition imposed by the remaining δ-function trivial to satisfy, it is convenient to change variables from $|p_1|$ to $E_1 + E_2$. The jacobian of this transformation is[1]:

$$\frac{d(E_1+E_2)}{d\,|p_1|} = \frac{\vec{p}_1}{|p_1|}\cdot\left(\frac{\vec{p}_1}{E_1}-\frac{\vec{p}_2}{E_2}\right).$$

We thus have

$$d\,|p_1| = d(E_1+E_2)\left(\frac{\vec{p}_1}{|p_1|}\cdot\left(\frac{\vec{p}_1}{E_1}-\frac{\vec{p}_2}{E_2}\right)\right)^{-1},$$

which gives

$$\delta(E_k-E_1-E_2)\frac{|p_1|^2\,d\,|p_1|\,d^2\Omega}{16\pi^2 E_{p_1}E_{p_2}} = \frac{1}{16\pi^2}\frac{|p_1|^3\,d^2\Omega}{\vec{p}_1\cdot(E_2\vec{p}_1-E_1\vec{p}_2)}.$$

If we specialize this result, valid in a general reference frame, to the centre-of-mass frame of the decaying massive particle, we have

$$E_1+E_2 = E_k = M_k\ ; \qquad \vec{p}_1 = -\vec{p}_2$$

in which case

$$\vec{p}_1\cdot(E_2\vec{p}_1-E_1\vec{p}_2) = E_k\,|p_1|^2\,.$$

In this particular frame, the differential decay rate is then

$$d\Gamma = \frac{1}{32\pi^2 M_k^2}\,|\mathcal{M}|^2\,|p_1|\,d(\cos\theta)\,d\phi\,, \tag{10}$$

where we recall energy-momentum conservation is implicit.

To evaluate eq. (9) in the C.M. frame of W^- and with $m_\nu = 0$, we recall:

$$\vec{p} = -\vec{q}; \qquad q_0 = |q|\ , \qquad p_0+q_0 = M_W\,; \qquad \epsilon\cdot p = -\epsilon\cdot q = |p|\cos\theta.$$

We can then workout the kinematics (which you should try to do by yourself), to find

$$p = \left(\frac{M_W^2+m_e^2}{2M_W}, \frac{m_e^2-M_W^2}{2M_W}, 0, 0\right)\ , \qquad q = \frac{M_W^2-m_e^2}{2M_W}(1,1,0,0).$$

With this explicit parametrisation, we can evaluate the dot products of eq. (9), and the result can then be inserted in eq. (10). Nothing depends on the angle ϕ, so it can be trivially integrated, giving a factor of 2π. We finally find

$$\frac{d\Gamma}{d\cos\theta} = \frac{e_W^2 M_W}{16\pi}(g_V^2+g_A^2)\left(1-\cos^2\theta\left(1-\frac{m_e^2}{M_W^2}\right)\right)\left(1-\frac{m_e^2}{M_W^2}\right)^2. \tag{11}$$

To obtain the unpolarised rate, we have two options. The first, and longest, is to average over the polarisations of W^-, as is done in chapter 4 of the book, using

$$\sum_{\lambda=-1,0,1}\epsilon_\mu(k,\lambda)\epsilon_\nu^*(k,\lambda) = \eta_{\mu\nu}+\frac{k_\mu k_\nu}{M_W^2}.$$

[1] We recall that when taking the derivative of a vector w.r.t. a length, we get a vector!

The second is simpler: having determined the linearly polarised rate, we can obtained the unpolarised rate by integrating over the angle θ (this angle was the parameter controlling the direction of the polarisation, by integrating over it we are averaging over all polarisations). The angle θ varies from 0 to π, which means its cosine should be integrated from -1 to 1. Computing this integral, we obtain

$$\Gamma(W^- \to e^- \bar{\nu}_e) = \int_{-1}^{1} \frac{d\Gamma}{d\cos\theta} d\cos\theta = \frac{e_W^2 M_W}{12\pi}(g_V^2 + g_A^2)\left(1 - \frac{m_e^2}{M_W^2}\right)^2 \left(1 + \frac{m_e^2}{2M_W^2}\right) . \quad (12)$$