Tutorial 2

November 2, 2015

Throughout the tutorials, we will use the conventions of *The Standard Model: A Primer* (Cliff Burgess and Guy Moore). In particular, we will use a diag$(-,+,+,+)$ metric, so that $x^2 = -x_0^2 + |\vec{x}|^2$ and $p^2 = -m^2 = -p_0^2 + |\vec{p}|^2$.

The Klein-Gordon field

A scalar free field $\varphi(x)$ is described by the so-called Klein-Gordon lagrangian density:

$$\mathcal{L}_{KG} = -\frac{1}{2}\partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2. \tag{1}$$

As in Classical Mechanics, the equation of motion corresponding to a lagrangian density $\mathcal{L}(\varphi, \partial_\mu \varphi)$ is obtained from

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \varphi\right)} \right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0. \tag{2}$$

Question 1: Show that the equation of motion for the Klein-Gordon lagrangian is

$$\left(\partial^2 - m^2\right) \varphi = 0. \tag{3}$$

Question 2: Show that the solution of eq. (3) for a real field $\varphi(x) = \varphi^*(x)$ is

$$\varphi(x) = \int \frac{d^3\vec{k}}{(2\pi)^3 2E_k} \left(a_{\vec{k}} e^{ik\cdot x} + a_{\vec{k}}^* e^{-ik\cdot x} \right). \tag{4}$$

As in Classical Mechanics, we can define the conjugate momentum

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x)}, \tag{5}$$

in terms of which we can define a hamiltonian density

$$\mathcal{H}(x) = \Pi(x)\dot{\varphi}(x) - \mathcal{L}(x), \tag{6}$$

where $\dot{\varphi} = \partial_0(\varphi)$.

Question 3: Use eq. (1) to show that

$$\Pi_{KG}(x) = \dot{\varphi}(x), \tag{7}$$

and then

$$\mathcal{H}_{KG}(x) = \frac{1}{2}\left(\Pi_{KG}^2 + (\nabla\varphi)^2 + m^2\varphi^2\right). \tag{8}$$

The hamiltonian is obtained from the hamiltonian density by integrating over the space

$$H = \int d^3\vec{x}\mathcal{H}(x). \tag{9}$$

Question 4: Use eq. (4) and eq. (8) to show that

$$H_{KG} = \frac{1}{2}\int \frac{d^3\vec{k}}{(2\pi)^3 2E_{\vec{k}}} E_{\vec{k}} \left(a^*_{\vec{k}}a_{\vec{k}} + a_{\vec{k}}a^*_{\vec{k}}\right). \tag{10}$$

Question 5: Use the commutation relations of bosonic particles studied in the first tutorial to argue that H_{KG} can be rewritten as

$$H_{KG} = \int \frac{d^3\vec{k}}{(2\pi)^3 2E_{\vec{k}}} E_{\vec{k}} a^*_{\vec{k}}a_{\vec{k}} + H_0, \tag{11}$$

where H_0 is an (infinite) constant quantity with no dynamic role.

Eq. (8) is called a quadratic hamiltonian (it is quadratic in the fields). A quadratic hamiltonian describe the evolution of free states, which means different states do not mix when evolving.

Question 6: Accordingly, show that

$$\langle\vec{q}|\, H_{KG}\, |\vec{p}\rangle = E_p \langle\vec{q}\,|\,\vec{p}\rangle\,. \tag{12}$$

Extra question, if the rest was very easy: As is done in the Lagrangian description of Classical Mechanics, show that eq. (2) can be obtained by solving $\delta S = 0$, where S is the action, defined as

$$S = \int d^4x\mathcal{L}(\varphi, \partial_\mu\varphi), \tag{13}$$

and δS is the variation of the action.