Tutorial 5

November 23, 2015

The Standard Model Lagrangian is invariant under the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge symmetry, and can be written as

$$\mathcal{L}_{SM} = \mathcal{L}_{fg} + \mathcal{L}_{\text{Higgs}}, \tag{1}$$

where $\mathcal{L}_{\text{Higgs}}$ contains all the terms involving the scalar field ϕ. We will not write explicitly each term because they are quite long (see e.g. *The Standard Model: A Primer* (Cliff Burgess and Guy Moore) for the full expression), but for this exercise we will recall the terms that will be necessary:

$$\begin{aligned}
\mathcal{L}_{fg} =& -\frac{1}{4}G^{\alpha}_{\mu\nu}G^{\alpha\mu\nu} - \frac{1}{4}W^{a}_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \\
& -\frac{1}{2}\bar{L}_m \not{D} L_m - \frac{1}{2}\bar{E}_m \not{D} E_m - \frac{1}{2}\bar{Q}_m \not{D} Q_m - \frac{1}{2}\bar{U}_m \not{D} U_m - \frac{1}{2}\bar{D}_m \not{D} D_m + \dots , \qquad (2) \\
\mathcal{L}_{\text{Higgs}} =& -\left(D_\mu\phi\right)^\dagger\left(D_\mu\phi\right) - V(\phi^\dagger\phi) + \dots , \qquad (3) \\
V(\phi^\dagger\phi) =& \lambda\left[\phi^\dagger\phi - \frac{\mu^2}{2\lambda}\right]^2 . \qquad (4)
\end{aligned}$$

The gauge field strengths are defined as usual in terms of the gauge bosons,

$$\begin{aligned}
G^{\alpha}_{\mu\nu} =& \partial_\mu G^{\alpha}_{\nu} - \partial_\nu G^{\alpha}_{\mu} + g_3 f^{\alpha}{}_{\beta\gamma} G^{\beta}_{\mu} G^{\gamma}_{\nu} \\
W^{a}_{\mu\nu} =& \partial_\mu W^{a}_{\nu} - \partial_\nu W^{a}_{\mu} + g_2 \epsilon_{abc} W^{b}_{\mu} G^{c}_{\nu} \\
B_{\mu\nu} =& \partial_\mu B_\nu - \partial_\nu B_\mu ,
\end{aligned} \tag{5}$$

associated respectively to the $SU(3)_c$, $SU(2)_L$ and $U(1)_Y$ components of the gauge group of the Standard Model.

The action of the covariant derivatives depends on which representation each matter field belongs to. For instance,

$$\begin{aligned}
D_\mu L_m =& \partial_\mu L_m + \left[\frac{i}{2}g_1 B_\mu - \frac{i}{2}g_2 W^{a}_{\mu}\tau_a\right]\mathrm{P_L} L_m + \left[-\frac{i}{2}g_1 B_\mu + \frac{i}{2}g_2 W^{a}_{\mu}\tau^{*}_{a}\right]\mathrm{P_R} L_m \\
D_\mu E_m =& \partial_\mu E_m + \left[i g_1 B_\mu\right]\mathrm{P_R} E_m + \left[-i g_1 B_\mu\right]\mathrm{P_L} E_m \\
D_\mu Q_m =& \partial_\mu Q_m + \left[-\frac{i}{6}g_1 B_\mu - \frac{i}{2}g_2 W^{a}_{\mu}\tau_a - \frac{i}{2}g_3 G^{\alpha}_{\mu}\lambda_\alpha\right]\mathrm{P_L} Q_m + \\
& \left[\frac{i}{6}g_1 B_\mu + \frac{i}{2}g_2 W^{a}_{\mu}\tau^{*}_{a} + \frac{i}{2}g_3 G^{\alpha}_{\mu}\lambda^{*}_{\alpha}\right]\mathrm{P_R} Q_m \\
D_\mu \phi =& \partial_\mu - \frac{i}{2}g_2 W^{a}_{\mu}\tau_a\phi - \frac{i}{2}g_1 B_\mu\phi ,
\end{aligned} \tag{6}$$

where τ_a are the three Pauli matrices (usually denoted σ, we use another notation because this is related to the $SU(2)_L$ gauge group), and λ_a are the 8 Gell-Mann matrices.

1 Representations of fields of the Standard Model

Question 1: From the expressions for $D_\mu L_m$, $D_\mu E_m$ and $D_\mu Q_m$, determine in which representations R_c of $SU(3)_c$ and R_L of $SU(2)_L$ the left and right components of the fermionic fields are, as well as the associated hypercharge Y. In other words, for the left and right components of L_m, E_m and Q_m, determine the 6 triplets

$$(R_c, R_L, Y)_{\mathrm{P_i} F_m} \tag{7}$$

for i = R, L and $F = L, E, Q$.

Question 2: Does the Higgs scalar couple to gluons? What is its $SU(2)_L$ representation and its hypercharge?

2 Higgs Lagrangian

We now focus on the part of the Standard Model Lagrangian which depends on the scalar field ϕ, $\mathcal{L}_{\text{Higgs}}$. ϕ is a complex field, so we can write it in terms of two real fields $|\phi|$ and ξ. Furthermore, it is a $SU(2)_L$ doublet and we can write it as (we will not attempt do justify why this is possible):

$$\phi = \begin{bmatrix} 0 \\ |\phi|\, e^{i\xi} \end{bmatrix} . \tag{8}$$

Question 3: Find the value $|\phi|_0$ that minimises the potential $V(\phi^\dagger\phi) = V(|\phi|^2)$.

Defining $v^2 = 2\,|\phi|_0^2$, we can write the Higgs scalar as

$$\phi = \phi_0 + H(x) = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{bmatrix} . \tag{9}$$

The field $H(x)$ is the Higgs particle recently discovered at the LHC.

Question 4: Write the Higgs potential in terms of the field $H(x)$. What is its mass?

3 Mass of gauge bosons

The component of the Standard Model Lagrangian independent of the scalar field has no term quadratic in the gauge boson fields. This would naïvely lead us to think that gauge bosons cannot be massive if they are to satisfy the gauge symmetry of the Standard Model. We now show this is not the case: the Higgs scalar ϕ breaks the $SU(2)_L \times U(1)_Y$ and gives a mass to (some of) the gauge bosons. Because the Higgs scalar does not couple to gluons, these drop out of the following discussion.

Question 5: Using eq. (9), show that

$$D_\mu \phi = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \partial_\mu H(x) \end{bmatrix} - \frac{i}{2\sqrt{2}} (v + H(x)) \begin{bmatrix} g_2 \left(W^1_\mu - i W^2_\mu\right) \\ g_1 B_\mu - g_2 W^3_\mu \end{bmatrix} . \tag{10}$$

We now examine how terms quadratic in the gauge bosons are generated in $(D_\mu \phi)^\dagger (D_\mu \phi)$.

Question 6: Show that

$$\begin{aligned} - (D_\mu \phi)^\dagger (D_\mu \phi) = & -\frac{1}{2} \partial_\mu H \partial^\mu H - \frac{1}{8} g_2^2 (v + H)^2 \left(W^1_\mu - i W^2_\mu\right) \left(W^{1\mu} + i W^{2\mu}\right) \\ & - \frac{1}{8} (v + H)^2 \left(g_1 B_\mu - g_2 W^3_\mu\right) \left(g_1 B^\mu - g_2 W^{3\mu}\right) , \end{aligned} \tag{11}$$

and isolate the quadratic terms in the fields B_μ and W^i_μ, with $i = 1, 2, 3$.

3.1 Mass of the W^1_μ and W^2_μ bosons

Question 7: From the first line of eq. (11), show that the mass of the W^1_μ and W^2_μ bosons is equal, and determine its expression in terms of v and g_2.

Note: For reasons we will not explain here, instead of the W^1_μ and W^2_μ bosons it is more convenient to use the combination

$$W^\pm_\mu = \frac{1}{\sqrt{2}} \left(W^1_\mu \mp i W^2_\mu\right) . \tag{12}$$

Note that the value we determined for the mass is still correct, $M_{W^\pm} = M_{W^1} = M_{W^2}$.

3.2 Mass of the B_μ and W^3_μ bosons

Unlike what happens for the W^1_μ and W^2_μ bosons, the second line of eq. (11) mixes the B_μ and W^3_μ bosons, and so we cannot simply read the mass of each boson. In fact, these fields are not appropriate to discuss mass terms. Clearly, the combination $g_1 B_\mu - g_2 W^3_\mu$ would be more suited. Let us then define

$$Z_\mu = \frac{g_2 W^3_\mu - g_1 B_\mu}{\sqrt{g_1^2 + g_2^2}} . \tag{13}$$

Question 8: Determine the mass of the Z_μ boson from the second line of eq. (11).

The Weinberg angle θ_W is defined as

$$\cos\theta_W = \frac{g_2}{\sqrt{g_1^2 + g_2^2}} , \qquad \sin\theta_W = \frac{g_1}{\sqrt{g_1^2 + g_2^2}} . \tag{14}$$

We can then rewrite

$$Z_\mu = W^3_\mu \cos\theta_W - B_\mu \sin\theta_W . \tag{15}$$

This means Z_μ can be seen as a rotation of the $\left(W^3_\mu, B_\mu\right)$ vector by an angle θ_W. The issue was that W^3_μ and B_μ were not mass eigenvectors, whereas Z_μ is and its mass is its eigenvalue.

Question 9: Construct the mass eigenvector orthogonal to Z_μ, which we call A_μ. For this, it might be useful to write the rotation of the vector

$$\begin{bmatrix} W_\mu^3 \\ B_\mu \end{bmatrix} \tag{16}$$

we described above in matrix form. What is the mass of this vector boson? Aside from the gluons, this is the only other massless vector boson of the Standard Model, the good old photon.

Question 10: $\sin^2\theta_W$ is an important quantity of the Standard Model. Experimentally, one finds

$$M_Z = 91,2GeV \qquad M_W = 80,4GeV\,. \tag{17}$$

Compute $\sin^2\theta_W$.

Let's recap what we have done so far: we started with four bosonic vector fields (the W_μ^i and the B_μ), for which no mass term could be included in the Standard Model Lagrangian because it wouldn't be invariant under the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge group. We then saw that through the breaking of the $SU(2)_L \times U(1)_Y$ symmetry induced by the scalar field ϕ we obtained three massive bosons, the $W_\mu^\pm$ and the Z_μ. This was good as we managed to give a mass to the gauge bosons, but we seemed to have lost a boson on the way. The fourth boson turns out to be the photon, which does not get a mass through the breaking of the symmetry, as we expected. We left out of the discussion how the fermionic matter fields acquire mass, but this is of course an important subject in which the Higgs scalar is again central.

Question 11 — if you have done everything else: Rewrite the covariant derivative for a fermion field of hypercharge Y and in a general $SU(2)_L$ representation in terms of the gauge bosons A_μ, Z_μ and $W_\mu^\pm$. Note that this requires taking specific combinations of the generators of $SU(2)_L$, consistently with the definition of the $W_\mu^\pm$ bosons in terms of the $W_\mu^{1,2}$. Under this form, the relation between the hypercharge and the electric charge should be explicit, and you should be able to write the electromagnetic coupling in terms of the g_1 and g_2 couplings.