Tutorial 8

December 21, 2015

Exercise 1

We start by reviewing some results about traces of gamma functions. Show that:

1) $\mathrm{tr}[\gamma^{\mu_1} \dots \gamma^{\mu_n}] = 0$ if n is odd.

2) $\mathrm{tr}[\gamma_5 \gamma^{\mu_1} \dots \gamma^{\mu_n}] = 0$ if n is odd.

3) $\mathrm{tr}[\gamma^\mu \gamma^\nu] = 4\eta^{\mu\nu}$

4) $\mathrm{tr}[\gamma_5 \gamma^\mu \gamma^\nu] = 0$

5) $\mathrm{tr}[\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho] = 4(\eta^{\mu\nu}\eta^{\lambda\rho} - \eta^{\mu\lambda}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\lambda})$

6) $\mathrm{tr}[\gamma_5 \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho] = 4i\epsilon^{\mu\nu\lambda\rho}$

To show some of these results, recall that the only invariant second-rank tensor is the metric. For 4) and 6), it is useful to check the symmetries of the expressions under the exchange of two of the indices. The above formulas use the definition $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ in terms of gamma-matrices with lower indices.

Exercise 2

Using the above results, evaluate

$$\mathrm{tr}\left[\gamma^\mu (g_V + g_A\gamma_5)(m_f + i\not{q})\gamma^\nu (g_V + g_A\gamma_5)(m_f - i\not{p})\right]. \tag{1}$$

Exercise 3

We will study the decay rate $\Gamma(W^- \to e^- \bar{\nu}_e)$. This decay is controlled by the Lagrangian,

$$\mathcal{L} = ie_W W_\mu^- \bar{e}\gamma^\mu (g_V + g_A\gamma_5)\nu + \mathrm{h.c.}. \tag{2}$$

Show that

$$\langle e(p)\bar{\nu}_e(q)|\mathcal{H}|W(k)\rangle = -ie_W \epsilon_\mu(k)\bar{u}(p)\gamma^\mu (g_V + g_A\gamma_5)v(q). \tag{3}$$

For $m_e \neq m_\nu \neq 0$, show that

$$\sum_{\sigma_1\sigma_2} |\epsilon_\mu(k)\bar{u}(p)\gamma^\mu(g_V + g_A\gamma_5)v(q)|^2 = 4\left[m_e m_\nu (g_V^2 - g_A^2) + (g_V^2 + g_A^2)(2\epsilon\cdot p\epsilon\cdot q - p\cdot q)\right], \tag{4}$$

Assume a basis of polarization vectors which is real valued; $\epsilon^\mu = \epsilon^{\mu *}$.

Compute the decay rate of a linearly polarized W^-. Assume vanishing neutrino masses $m_\nu = 0$ and work in the rest frame of the W bosons. For W bosons polarized in the $\vec{e}$ direction one obtains,

$$\frac{d\Gamma}{d\cos\theta} = \frac{e_W^2 M_W}{16\pi}(g_V^2 + g_A^2)\left(1 - \cos^2\theta\left(1 - \frac{m_e^2}{M_W^2}\right)\right)\left(1 - \frac{m_e^2}{M_W^2}\right)^2, \tag{5}$$

where θ is the angle between the electron momentum $\vec{p}$ and the polarization axis $\vec{e}$.

Finally, consider the decay rate of unpolarized W-bosons in their rest frame. Show that the decay rate is given by,

$$\Gamma(W^- \to e^- \bar{\nu}_e) = \frac{e_W^2 M_W}{12\pi}(g_V^2 + g_A^2)\left(1 - \frac{m_e^2}{M_W^2}\right)^2 \left(1 + \frac{m_e^2}{2M_W^2}\right) . \tag{6}$$

This result can be obtained by summing over the polarization states or by appropriate averaging the polarized result. It is instructive to compare both approaches.

Remark:

While it is important to do the above algebra by hand, it is recommended to use as well the Mathematica package FeynCalc (`www.feyncalc.org`) for cross checks.